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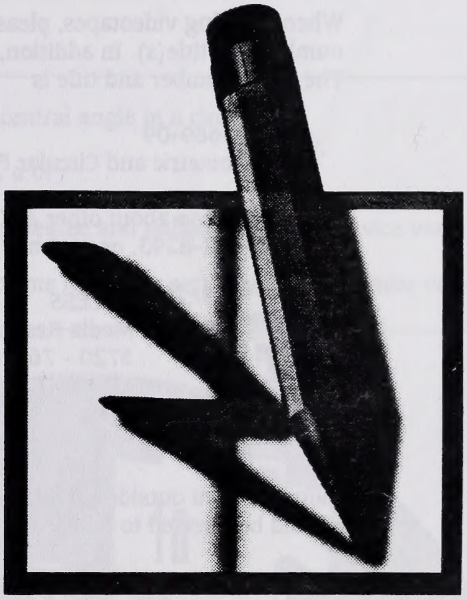
MATH FACTOR

TRIGONOMETRIC AND CIRCULAR FUNCTIONS



Lesson 1: Radian Measurement

Student Workbook

Produced by

THE EDUCATION STATION

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This guide accompanies **Radian Measurement** from the Math Factor series.

Writer: Wendy Stanhope
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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Trigonometric and Circular Functions: Radian Measurement

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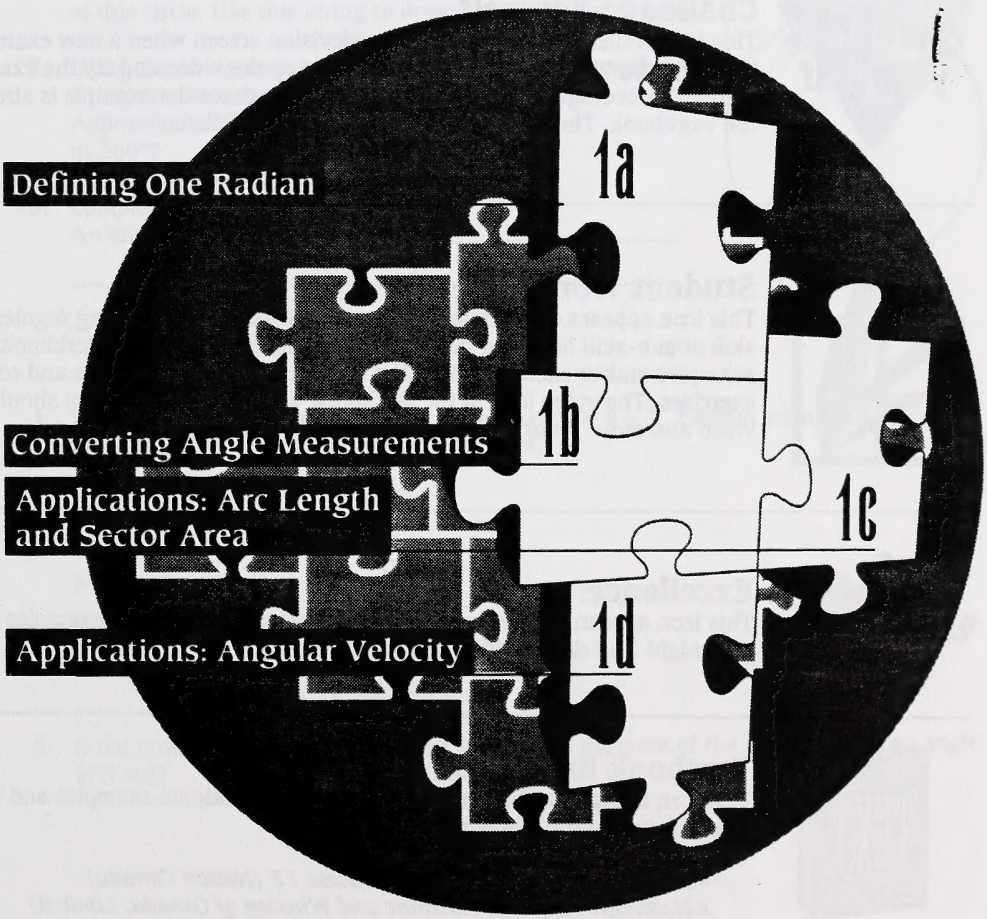
Lesson 1: RADIAN MEASUREMENT

LEARNER EXPECTATIONS

By the end of Lesson 1 you should be able to:

- define the radian measurement of a central angle in a circle
- use the formula $\theta = \frac{a}{r}$ to solve for θ, a or r
- convert angle measurements between degree and radian measure and vice versa
- apply this knowledge to solving problems involving sector area and angular velocity

This lesson has four parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

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- challenge yourself to find the solution before it is given
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Textbook Reference:

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- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Defining One Radian

MATH FACTOR

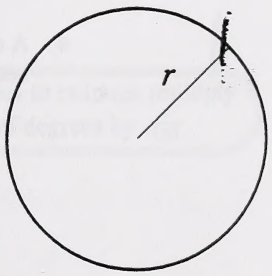
Watch the Introduction and Video Segment 1a.

TEXT

Additional Examples
Mathematics 12: pp. 198, 199
Mathematics: Principles and Process 12: p. 181
Holtmath 12: p. 287

Exercises

1. Cut off a piece of string that is equal in length to the radius of this circle. Use this string to draw an angle of one radian.



2. Using the string as a measuring tool, how many lengths of string are there around the circle's circumference? Approximately how many radians are there in 360°?

3. Complete the following definition.
An angle of one radian is defined as _____
_____
_____
_____

4. In general, radians can be defined by the formula $\theta = \frac{a}{r}$. Explain what each of the following symbols represents in the formula.

   a. θ

   b. a

   c. r

5. If the radius of a circle is 7 cm, then what is the measure of the arc length for an angle of 5 rad?

6. The central angle of a circle of radius 10 cm subtends an arc length of 45 cm. What is the radian measure of this central angle?

7. The central angle of a circle of radius r cm subtends an arc length of $2\pi r$. What is the radian measure of this central angle?

8. A circle of radius r has a circumference of $2\pi r$. What is the degree measure of the central angle that subtends an arc length of $2\pi r$?

9. Compare your answers in questions 7 and 8. What degree measure is equivalent to 2π rad?

EXERCISE 1a SOLUTIONS are at the back of this workbook.

Converting Angle Measurements

Challenge yourself

Convert -112° to radian measure.

Watch Video Segment 1b.

Example 1

Convert 60° to radian measure.

Solution

Method 1

$180^\circ = \pi$ rad

$1^\circ = \frac{\pi}{180}$ rad

$60^\circ = 60^\circ\left(\frac{\pi}{180^\circ}\right)$ rad

$= \frac{\pi}{3}$ rad

To convert degrees to radians, multiply the number of degrees by $\frac{\pi}{180}$.

Method 2

$\frac{\pi}{180^\circ} = \frac{n}{60^\circ}$

$180^\circ n = 60^\circ \pi$

$n = \frac{60^\circ \pi}{180^\circ}$

$= \frac{\pi}{3}$

Make equivalent ratios of radians to degrees: $\frac{\pi}{180^\circ} = \frac{\text{number of radians}}{\text{number of degrees}}$ or using $\frac{180^\circ}{\pi}$, you can compare degrees to radians.

Exact Radian Measure

$\frac{\pi}{3}$ is the exact radian equivalent for 60°.

For exact values, retain the symbol π.

Approximate Radian Measure

$60^\circ = \frac{\pi}{3}$

$\doteq \frac{3.14}{3}$

$\doteq 1.05$ rad

To express radians in decimal form, replace π by 3.14... by using the π button on your calculator.

Example 2

Convert $\frac{3}{4}\pi$ rad to degree measure.

Solution

$$\pi \text{ rad} = 180^\circ$$

$$\frac{3}{4}\pi \text{ rad} = \frac{3}{4}\left(180^\circ\right)$$

$$= 135^\circ$$

To convert exact radians to degrees, replace π rad by 180°.

Example 3

Convert 3.66 rad to degree measure.

Solution

Method 1

$$1 \text{ rad} = \frac{180^\circ}{\pi}$$

$$3.66 \text{ rad} = 3.66\left(\frac{180^\circ}{\pi}\right)$$

$$\doteq 210^\circ$$

To convert decimal equivalent radians to degree measure, multiply the number of radians by $\frac{180^\circ}{\pi}$, where $\pi = 3.14\ldots$.

Method 2

degrees $\longrightarrow \frac{180^\circ}{\pi} = \frac{n}{3.66} \longleftarrow$ degrees
radians $\longrightarrow$ radians $\longleftarrow$ radians

Make equivalent ratios of degrees to radians or radians to degrees.

$$180^\circ\left(3.66\right) = n\pi$$

Solve for n.

$$\frac{180^\circ\left(3.66\right)}{\pi} = n$$

$$\frac{180^\circ\left(3.66\right)}{3.14} \doteq n$$

Using the π button on a calculator, $\pi = 3.14\ldots$.

$$n \doteq 210^\circ$$

TEXT

Additional Examples
Mathematics 12: p. 200 (Question 3)
Mathematics: Principles and Process 12: pp. 181, 182
Holtmath 12: p. 287

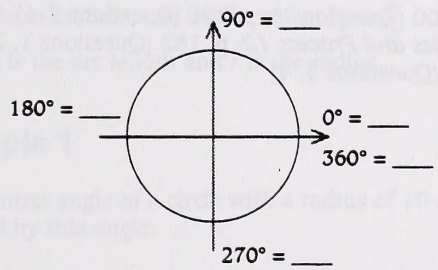
Exercises

10. Complete the following table.

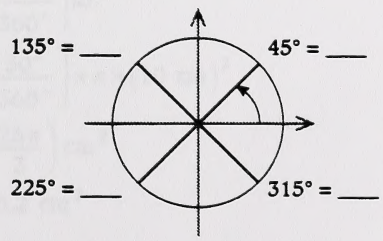
Degrees	Exact Radian Measure	Approximate Radian Measure
0°		
	$\frac{\pi}{2}$	
	π	3.14
270°		
	2π	6.28
	$\frac{\pi}{6}$	
		5.76
225°		
	$\frac{5\pi}{3}$	

11. Certain angles are identified on each of the following circles. Convert the measure of the identified angles from degree to exact radian measure.

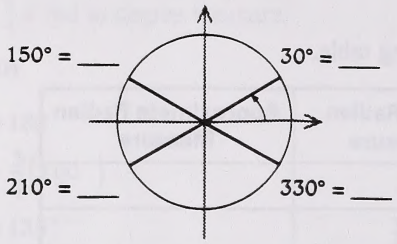
a. quadrant angles



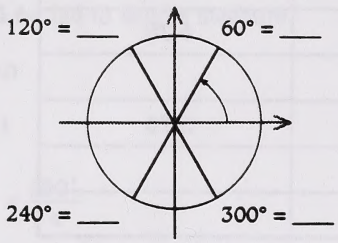
b. angles with a reference angle of 45°



c. angles with a reference angle of 30°



d. angles with a reference angle of 60°



EXERCISE 1b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 200 (Question 1), p. 201 (Questions 2-4)
Mathematics: Principles and Process 12: p. 182 (Questions 1, 2, 5), p. 183 (Question 9)
Holtmath 12: p. 287 (Questions 3, 4)

Applications: Arc Length and Sector Area

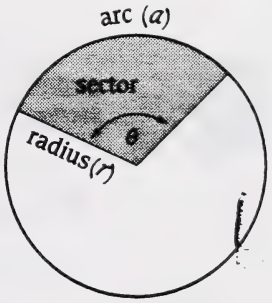
Challenge yourself

Find the radian measurement of a central angle with an arc length of 40 cm if the radius of the circle is 12 cm.

MATH FACTOR

Watch Video Segment 1c.

A sector of a circle is the region bounded by a central angle of a circle and the arc it intercepts. The area of a sector is dependent on the area of the circle and the measure of the central angle.



To find the area of a circle, use the formula $A = \pi r^2$.

To find the area of a sector when the measure of angle θ is in degrees, use the formula $A_S = \left(\frac{\theta}{360^\circ}\right)\pi r^2$.

To find the area of a sector when the measure of angle θ is in radians, use the formula $A_S = \frac{ar}{2}$, where a is the arc length and r is the radius.

Since $\theta = \frac{a}{r}$, $a = \theta r$.
(θ is in radians.)

Example 1

If the central angle of a circle with a radius of 10 cm is 30°, then find the area of the sector enclosed by this angle.

Solution

$$A_S = \left(\frac{\theta}{360^\circ}\right)\pi r^2$$
$$= \left(\frac{30^\circ}{360^\circ}\right) \times \pi \times (10\text{ cm})^2$$
$$= \left(\frac{25\pi}{3}\right)\text{cm}^2$$
$$\doteq 26.2\text{ cm}^2$$

Example 2

If the central angle of a circle with a radius of 10 cm is $\frac{\pi}{6}$, then find the area of the sector enclosed by this angle.

Solution

$$A_S = \frac{ar}{2} \quad \text{or} \quad A_S = \frac{(\theta r)r}{2}$$

$$= \frac{\left(\frac{\pi}{6} \times 10 \text{ cm}\right) \times 10 \text{ cm}}{2}$$

$$= \frac{25\pi}{3} \text{ cm}^2$$

Additional Examples
Mathematics 12: p. 200 (Example 4)
Holtmath 12: p. 288 (Example a)

Exercises

12. If the radius of a circle is 1 cm, then find the arc length of a central angle of 210°.

13. If the central angle of a circle with a radius of 4 cm is 135°, then find the area of the sector enclosed by this angle.

14. If the central angle of a circle with a radius of 2 cm is $\frac{5\pi}{3}$, then find the area of the sector enclosed by this angle.

EXERCISE 1c SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 201 (Questions 5-7, 9, 10), p. 202 (Questions 12, 14)
Holtmath 12: p. 288 (Questions 1, 2, 4)

Applications: Angular Velocity

If the diameters of the wheel of an adult bicycle and a child's bicycle are 69 cm and 58 cm respectively, then find the angular velocities for an adult and a child who both ride their bikes a distance of 40 m in 10 s.

Watch Video Segment 1d.

When an object rotates around a central point, its velocity is called angular velocity. It is determined by the following formula:

$w = \frac{\theta}{t}$, where w = angular velocity
θ = angle of rotation measured in radians
t = time elapsed in appropriate units

Example 1

The angular velocity of a flywheel is 150 rev/min. Express this angular velocity in rad/s.

Solution

$$\frac{150 \text{ rev}}{1 \text{ min}} \times \frac{1 \text{ min}}{60 \text{ s}} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} = \frac{5\pi \text{ rad}}{\text{s}}$$

Each revolution is equal to 2π rad.

Example 2

If the radius of the flywheel in Example 1 is 12 cm, then determine the distance travelled by a point on the circumference of the flywheel in 10 s.

Solution

$$\theta = \left(\frac{5\pi \text{ rad}}{\text{s}}\right) \times (10 \text{ s})$$
$$= 50\pi$$

The flywheel wheel has rotated through 50π rad in 10 s.

$$\theta = \frac{a}{r}$$
$$50\pi = \frac{a}{12}$$
$$a = 12 \text{ cm} \times (50\pi)$$
$$= 600\pi \text{ cm}$$

Additional Examples
Mathematics: Principles and Process 12: p. 184
Holtmath 12: p. 288

Exercises

15. The radius of a wheel on a car is 32 cm. If the car is travelling at 120 km/h, what is the angular velocity of the tire in rad/s?

EXERCISE 1d SOLUTIONS are at the back of this workbook.

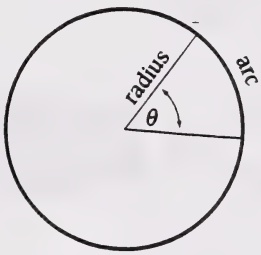
TEXT

Additional Practice
Mathematics 12: p. 202 (Question 13)
Mathematics: Principles and Process 12: p.184 (Questions 13-18)
Holtmath 12: p. 288 (Questions 3, 6)

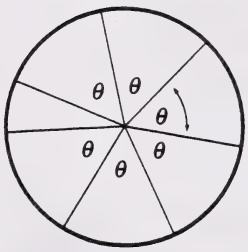
WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. Radian measurements are used throughout the remaining lessons of this module. Therefore, you should be skillful with expressing angles in radians before you continue on to **Lesson 2: Development of Circular Functions**.

Solutions

EXERCISE 1a

1. 

$\theta = 1$ rad

2. 

There are about 6 rad $(6.28...)$ in 360°.

3. An angle of one radian is defined as **the angle that is subtended by an arc of measure r on a circle whose radius is length r.**

4. a. θ represents the angle measured in radians. (Degrees cannot be used in this formula.)

   b. a represents the arc length (a distance along the circumference of the circle).

   c. r represents the radius.

5. $\theta = \frac{a}{r}$

   $a = \theta r$

   $= 5 \times 7$

   $= 35$ cm

6. $\theta = \frac{a}{r}$

   $= \frac{45 \text{ cm}}{10 \text{ cm}}$

   $= 4.5$ rad

7. $\theta = \frac{a}{r}$

   $= \frac{2\pi r}{r}$

   $= 2\pi$ rad

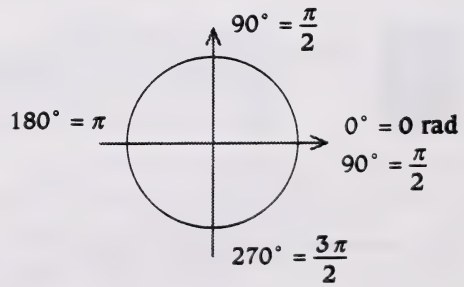
8. 360°

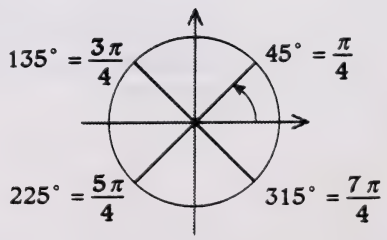
9. 2π rad = 360°

EXERCISE 1b

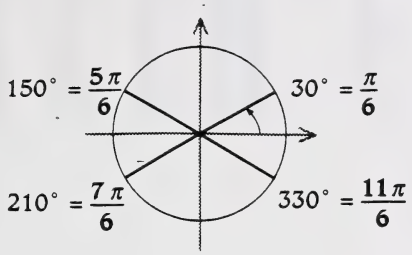
10.

Degrees	Exact Radian Measure	Approximate Radian Measure
0°	0	0.00
90°	$\frac{\pi}{2}$	1.57
180°	π	3.14
270°	$\frac{3\pi}{2}$	4.71
360°	2π	6.28
30°	$\frac{\pi}{6}$	0.52
330°	$\frac{11\pi}{6}$	5.76
225°	$\frac{5\pi}{4}$	3.93
300°	$\frac{5\pi}{3}$	5.24

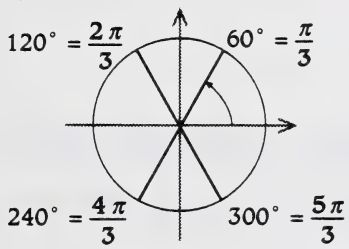
11. a. 

b. 

c.



d.



EXERCISE 1c

12. $210° = 210°\left(\frac{\pi}{180°}\right)$

$= \frac{7\pi}{6}$ rad

$\theta = \frac{a}{r}$

$a = \theta r$

$= \frac{7\pi}{6} \times 1$

$= \frac{7\pi}{6}$ cm

13. $A = \left(\frac{\theta}{360°}\right) \times \pi r^2$

$= \left(\frac{135°}{360°}\right) \times \pi \times (4 \text{ cm})^2$

$= 6\pi \text{ cm}^2$

$\doteq 18.8 \text{ cm}^2$

14. $A_S = \frac{ar}{2}$

$= \frac{(\theta r)r}{2}$

$= \frac{\frac{5\pi}{3} \times 2 \times 2}{2}$ (since $a = \theta r$)

$= \frac{10\pi}{3} \text{ cm}^2$

$\doteq 10.5 \text{ cm}^2$

EXERCISE 1d

15. Find the circumference of the tire.

$C = 2\pi r$

$= 2\pi(32)$

$= 64\pi$ cm

$\doteq 201$ cm

Convert 120 km/h to cm/s.

$$\frac{120 \text{ km}}{1 \text{ h}} \times \frac{100\,000 \text{ cm}}{1 \text{ km}} \times \frac{1 \text{ h}}{3600 \text{ s}} \doteq \frac{3333 \text{ cm}}{\text{s}}$$

Find the number of tire revolutions in 1 s.

$$\frac{3333 \text{ cm}}{\text{s}} \div 201 \text{ cm} = \frac{16.58 \text{ rev}}{\text{s}}$$

Each revolution equals 2π rad.

$$\therefore \text{angular velocity} = \frac{16.58 \text{ rev}}{\text{s}} \times \frac{2\pi \text{ rad}}{\text{rev}}$$

$$\doteq \frac{104 \text{ rad}}{\text{s}}$$

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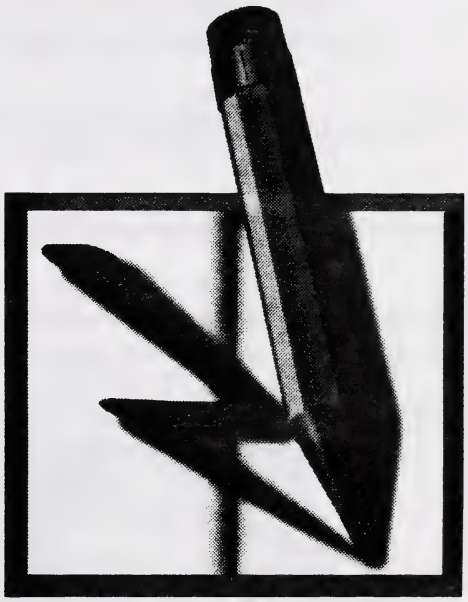

Lesson 2:
Development of Circular Functions

Student Workbook

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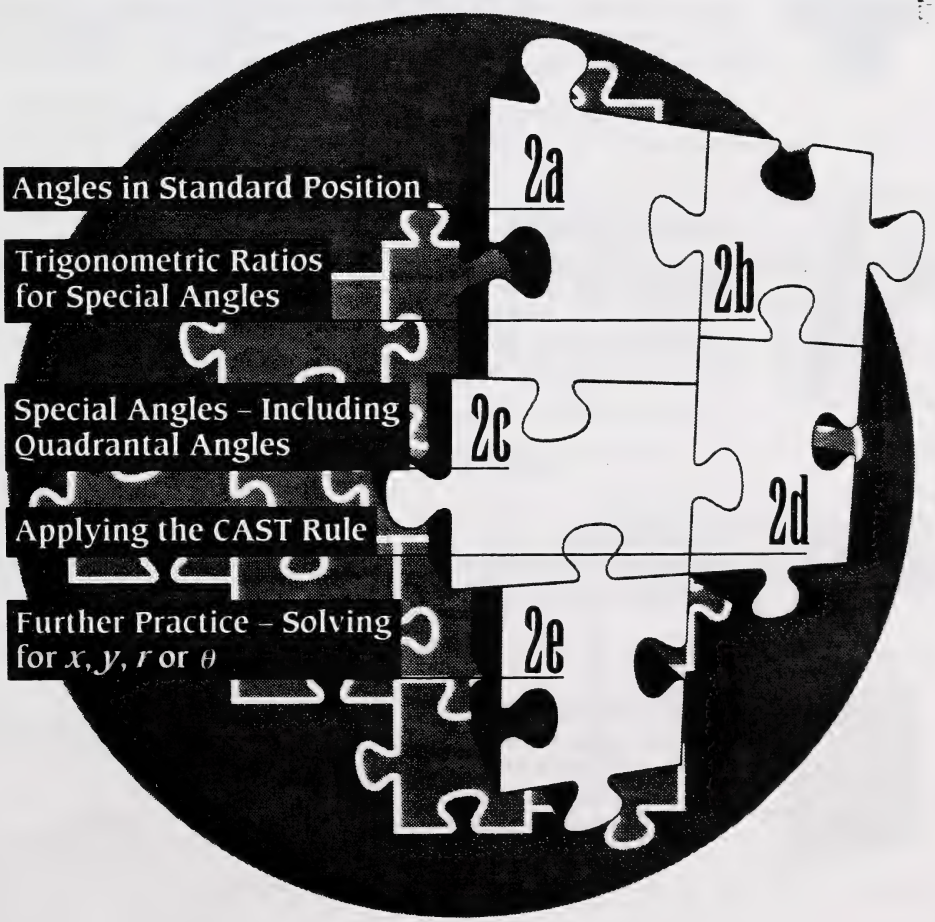
Lesson 2: DEVELOPMENT OF CIRCULAR FUNCTIONS

LEARNER EXPECTATIONS

By the end of Lesson 2 you should be able to:

- recognize and sketch positive and negative angles (measured in radians) in standard position
- determine the reference angle for any angle, θ (measured in radians) drawn in standard position
- determine the exact values of the trigonometric ratios for angles coterminal with $\frac{n\pi}{6}$, $\frac{n\pi}{3}$, $\frac{n\pi}{4}$, $\frac{n\pi}{2}$ and $n\pi$, $n \in \mathrm{I}$
- solve for x, y, r or θ using any trigonometric ratio for any angle, θ, in standard position, where r is the length of the terminal arm and (x, y) are the coordinates of a point, P, on the terminal arm

This lesson has five parts:



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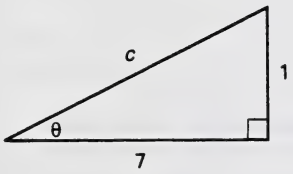
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Building Blocks

Do you remember how to use the sine, cosine and tangent functions as well as their reciprocal functions to solve for unknown angles or sides of right triangles? Before watching this program, refresh your memory by studying the following examples and doing the exercises.

Example 1

Determine the exact values of the six trigonometric ratios for this right triangle.



Primary Ratios

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

Reciprocal Ratios

$$\text{cosecant }\theta = \frac{\text{hypotenuse}}{\text{opposite side}}$$

$$\text{secant }\theta = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

$$\text{cotangent }\theta = \frac{\text{adjacent side}}{\text{opposite side}}$$

Solution

Find the length of the hypotenuse using the Pythagorean Theorem.

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 1^2 + 7^2 \\ &= 50 \\ c &= \sqrt{50} \\ &= 5\sqrt{2} \end{aligned}$$

To express $\sqrt{50}$ as a mixed radical, take the square root of any factors that are perfect squares.

$$\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$$

Now list the exact values of the six trigonometric ratios for angle θ.

Primary Ratios

$$\sin\theta = \frac{1}{5\sqrt{2}} = \frac{\sqrt{2}}{10}$$

$$\cos\theta = \frac{7}{5\sqrt{2}} = \frac{7\sqrt{2}}{10}$$

$$\tan\theta = \frac{1}{7}$$

Reciprocal Ratios

$$\csc\theta = \frac{5\sqrt{2}}{1}$$

$$\sec\theta = \frac{5\sqrt{2}}{7}$$

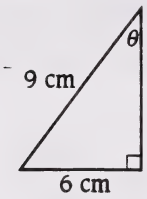
$$\cot\theta = \frac{7}{1}$$

To **rationalize** the denominator is to express the denominator as a rational number. An example follows:

$$\sin\theta = \frac{1}{5\sqrt{2}} = \frac{1 \times \sqrt{2}}{5\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{5 \times 2} = \frac{\sqrt{2}}{10}$$

Example 2

Find the measure of θ in degrees and radians. Write your answer correct to the nearest tenth.



Solution

$$\sin\theta = \frac{6\text{ cm}}{9\text{ cm}}$$

$$\sin\theta \doteq 0.6^\circ$$

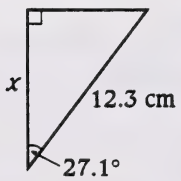
$$\theta \doteq 41.8^\circ \text{ or } \theta \doteq 0.7 \text{ rad}$$

To find θ in degrees, put your calculator in degree mode. Use the second function of the sine button to find the unknown angle.

To find θ in radians, put your calculator in radian mode; you can also solve for θ in degrees and switch your answer to radians by multiplying by $\frac{\pi}{180^\circ}$.

Example 3

Find the length of x. Write your answer correct to the nearest tenth.



Solution

$$\cos 27.1^\circ = \frac{x}{12.3\text{ cm}}$$

$$x \doteq 12.3\left(\cos 27.1^\circ\right)$$

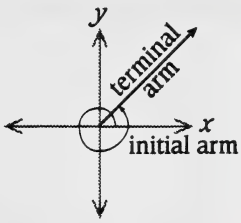
$$\doteq 10.9\text{ cm}$$

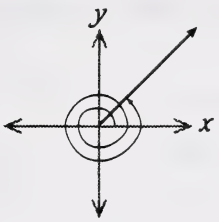
Since the angle is in degrees, be sure that your calculator is in degree mode. Enter cos 27.1°, then multiply by 12.3.

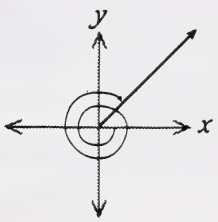
Example 4

Angles in standard position where the terminal arm is the same are called **coterminal angles**. Name the smallest positive coterminal angle for each angle, θ.

$\theta = 420°$ $\quad \theta = \frac{13\pi}{3}$ $\quad \theta = \frac{-11\pi}{3}$







Solution

$420° - 360° = 60°$

$\frac{13\pi}{3} - 2\pi = \frac{7\pi}{3}$

$\frac{7\pi}{3} - 2\pi = \frac{\pi}{3}$

or $\frac{13\pi}{3} - 4\pi = \frac{\pi}{3}$

$-\frac{11\pi}{3} + 2\pi = -\frac{5\pi}{3}$

$-\frac{5\pi}{3} + 2\pi = \frac{\pi}{3}$

or $\frac{-11\pi}{3} + 4\pi = \frac{\pi}{3}$

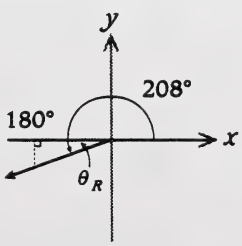
All of the preceding angles are coterminal with a 60° angle ($\frac{\pi}{3} = 60°$). The smallest positive coterminal angle is always between 0° and 360° (or 0 rad and 2π rad).

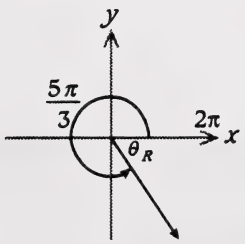
Example 5

State the reference angle, θ_R, for each of the following angles of rotation, θ.

$\theta = 208°$ $\quad \theta = \frac{5\pi}{3}$

Solution





A reference angle is the angle formed between the terminal arm of a given angle and the x-axis. The reference angle is a positive acute angle.

$\theta_R = 208° - 180°$
$= 28°$

$\theta_R = 2\pi - \frac{5\pi}{3}$
$= \frac{\pi}{3}$

Additional Examples
Mathematics 12: pp. 204, 210
Mathematics: Principles and Process 12: pp. 180, 188, 235, 236
Holtmath 12: pp. 272, 273, 275, 276, 278 (Example 2), pp. 284, 289

Exercises

1. Find the values of m and n using the diagrams; then state the exact values of the following:

   a. $\sin 30° =$ _____

   b. $\cos 30° =$ _____

   c. $\tan 30° =$ _____

   d. $\sin 60° =$ _____

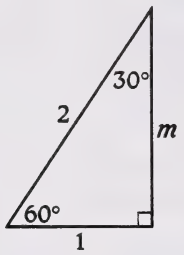
   e. $\cos 60° =$ _____

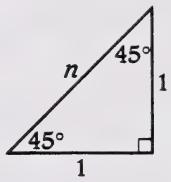
   f. $\tan 60° =$ _____

   g. $\sin 45° =$ _____

   h. $\cos 45° =$ _____

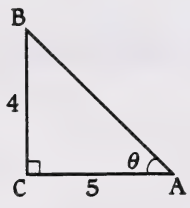
   i. $\tan 45° =$ _____



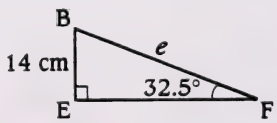


2. Find the indicated values for the following. Round your answers to the nearest tenth.

   a. Find the measure of θ in degrees.



   b. Find the measure of e in centimetres.

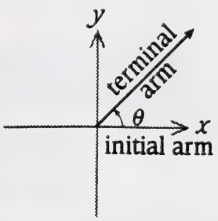


3. For each angle, θ, state the smallest positive coterminal angle and draw each angle in standard position.

   a. $\theta = 675°$

   b. $\theta = \frac{10\pi}{3}$

   c. $\theta = -\frac{5\pi}{4}$



This angle is in **standard position**. The vertex is at the origin. The initial arm is always on the positive x-axis. The terminal arm can be in any quadrant.

4. State the reference angle, θ_R, for each of the following:

   a. $\theta = 675°$

   b. $\theta = \frac{10\pi}{3}$

   c. $\theta = -\frac{5\pi}{4}$

BUILDING BLOCKS SOLUTIONS are at the back of this workbook.

Additional Practice

Mathematics 12: p. 207 (Questions 1, 2), p. 211 (Questions 1, 3, 4, 6, 7)

Mathematics: Principles and Process 12: p. 179 (Question 1), p. 180 (Questions 9-13),
p. 182 (Questions 3, 4),
p. 183 (Questions 6, 8, 10, 12),
p. 186 (Questions 1, 2),
p. 189 (Questions 1, 3-6),
p. 190 (Questions 9-11),
p. 236 (Questions 1-5)

Holtmath 12: p. 274 (Questions 2, 3, 5, 7, 9, 10), p. 277 (Questions 3, 6-8, 10),
p. 279 (Question 4), p. 282 (Questions 3, 6, 8), p. 285 (Question 6),
p. 286 (Questions 8, 9), p. 287 (Question 5), p. 295 (Questions 3, 11)

Note: If you are experiencing difficulty doing these practice exercises, you should review your Mathematics 20 Trigonometry Unit.

Angles in Standard Position

MATH FACTOR

Watch Video Segment 2a.

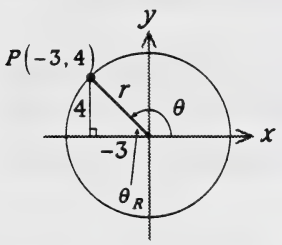
Example 1

Given $P(-3, 4)$ on the terminal arm of an angle, θ, find the exact values of the six trigonometric values for angle θ.

Solution

Draw θ in standard position. Draw the reference triangle by drawing a vertical arm from the end point of the terminal arm to a point on the x-axis. θ is the **angle of rotation** and θ_R is the **reference angle**.

Recall that you use the absolute value of the length of the legs of the right triangle to solve for θ_R. To solve for the radius, use the **Pythagorean Theorem**.



$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (3)^2 + (4)^2 \\ &= 9 + 16 \\ &= 25 \\ r &= 5 \end{aligned}$$

$$\therefore \sin\theta = \frac{4}{5} \qquad \csc\theta = \frac{5}{4}$$

$$\cos\theta = -\frac{3}{5} \qquad \sec\theta = -\frac{5}{3}$$

$$\tan\theta = -\frac{4}{3} \qquad \cot\theta = -\frac{3}{4}$$

In this example, $90° < \theta < 180°$, four of the six trigonometric ratios have negative values. Only the sine ratio and its reciprocal ratio, cosecant, are positive. Using the absolute value of any of the trigonometric ratios, you can solve for the reference angle. $\theta_R = 53°$ $\theta = 180° - 53° = 127°$

TEXT

Additional Examples
Mathematics 12: pp. 203, 209
Mathematics: Principles and Process 12: pp. 172, 173
Holtmath 12: pp. 280, 281

Exercises

1. The point $P(3,-2)$ is on the terminal arm of θ.

   a. Draw θ in standard position.

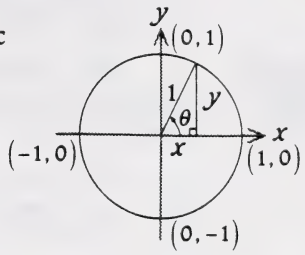
   b. Determine the exact values of the primary and reciprocal trigonometric ratios of θ.

   c. Determine the measure of θ_R and θ to the nearest degree.

   d. Determine the measure of θ in radians, expressed to the nearest tenth.

   e. Explain why only the cosine and secant ratios are positive in this question.

2. A circle with a radius of one is called a **unit circle**. Determine the primary and reciprocal trigonometric ratios for θ on the unit circle.



3. If $P(x, y)$ is the end point of the terminal arm of angle θ, then state the conditions for which $\cos\theta = x$ and $\sin\theta = y$.

EXERCISE 2a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 207 (Question 3), p. 208 (Questions 4, 6)
Mathematics: Principles and Process 12: p. 174 (Questions 1-4), p. 186 (Question 3), p. 187 (Question 4)
Holtmath 12: p. 282 (Questions 9, 10), p. 283 (Questions 11, 12)

Trigonometric Ratios for Special Angles

Challenge yourself

Complete the table by filling in the exact values for each of the trigonometric ratios of the angles indicated.

θ	sin θ	cos θ	tan θ
$\frac{\pi}{3}$			
$\frac{\pi}{6}$			
$\frac{\pi}{4}$			

MATH FACTOR

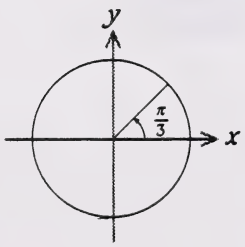
Watch Video Segment 2b.

Example 1

Determine all of the positive rotation angles, θ, that have a reference angle, θ_R, of $\frac{\pi}{3}$ for $0 \le \theta \le 2\pi$.

Solution

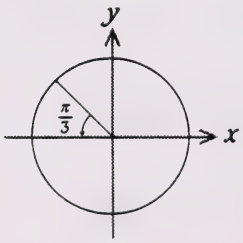
Quadrant 1



Rotation Angle = Reference Angle

$\therefore \theta = \theta_R$

Quadrant 2

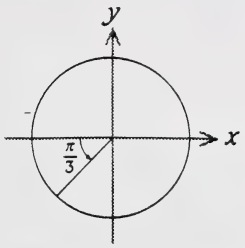


Rotation Angle = 1π – Reference Angle

$$\theta = 1\pi - \frac{\pi}{3}$$

$$= \frac{2\pi}{3}$$

Quadrant 3

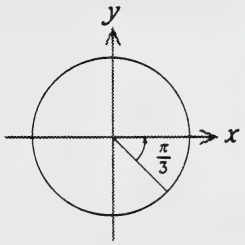


Rotation Angle = 1π + Reference Angle

$$\theta = \pi + \frac{\pi}{3}$$

$$= \frac{4\pi}{3}$$

Quadrant 4



Rotation Angle = 2π – Reference Angle

$$\theta = 2\pi - \frac{\pi}{3}$$

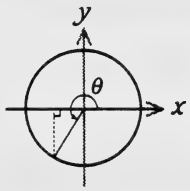
$$= \frac{5\pi}{3}$$

Example 2

Determine the exact value of $\cos \frac{4\pi}{3}$.

Solution

First sketch the angle.



Then find the reference angle of θ.

$$\theta = \frac{4\pi}{3}$$

$$= \pi + \theta_R$$

$$\theta_R = \theta - \pi$$

$$= \frac{4\pi}{3} - \pi$$

$$= \frac{\pi}{3}$$

Finally,

$$\cos \theta = \frac{x}{r}$$

$$\cos \frac{\pi}{3} = \frac{1}{2}$$

$$\therefore \cos \frac{4\pi}{3} = -\frac{1}{2}$$

TEXT

Additional Examples
Mathematics 12: pp. 212, 213, 214 (Example 2)
Mathematics: Principles and Process 12: pp. 185, 186
Holtmath 12: p. 289, p. 294 (Example 1)

Exercises

4. Determine all of the positive angles, θ, that have a reference angle of θ_R.

   a. $\theta_R = \frac{\pi}{6}$, for $0 \le \theta < 2\pi$

When the domain, $0 \le \theta < 2\pi$, is given in radians, the solutions must also be in radians.

   b. $\theta_R = \frac{\pi}{4}$, for $0 \le \theta < 2\pi$

5. Determine the reference angle, θ_R, for each of the following angles of rotation, θ and sketch θ in standard position.

   a. $\theta = \frac{7\pi}{6}$, $\theta_R =$ _____

b. $\theta = \frac{5\pi}{4}$, $\theta_R =$ _____

c. $\theta = \frac{11\pi}{3}$, $\theta_R =$ _____

d. $\theta = -\frac{5\pi}{4}$, $\theta_R =$ _____

6. Determine the exact value of each of the following:

a. $\sin \frac{23\pi}{6}$

b. $\tan \frac{7\pi}{4}$

EXERCISE 2b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 215 (Questions 1, 2)
Mathematics: Principles and Process 12: p. 187 (Questions 5-7)
Holtmath 12: p. 290 (Questions 4-6), p. 295 (Question 8)

Special Angles — Including Quadrantal Angles

Challenge yourself

Determine the exact values of the following:

a. $\sin 330^\circ$

b. $\sec \frac{19\pi}{4}$

c. $\sin \frac{\pi}{2}$

MATH FACTS

Watch Video Segment 2c.

Example 1

Determine the primary trigonometric ratios for $\theta = \pi$.

Solution

The terminal arm of θ is on the x-axis, therefore $x = r$. The end point, P, has coordinates $(x, 0)$ on the terminal arm.

$\sin \theta = \frac{y}{r}$ $\qquad$ $\sin \pi = \frac{0}{r} = 0$

$\cos \theta = \frac{x}{r}$ $\qquad$ $\cos \pi = \frac{x}{r} = -1$

$\tan \theta = \frac{y}{x}$ $\qquad$ $\tan \pi = \frac{0}{x} = 0$

Since $x = r$, $x < 0$, and $r > 0$.

To make these calculations even easier, you may wish to use the **unit circle** (a circle with a radius of one). Then $P(x, 0)$ becomes $P(-1, 0)$, since $x = -1$.

Example 2

Determine the exact value for $\csc\left(-\frac{5\pi}{4}\right)$.

Solution

Find the positive coterminal angle.

$$-\frac{5\pi}{4} + 2\pi = \frac{3\pi}{4}, \therefore \theta = \frac{3\pi}{4}$$

Find the reference angle.

$$\theta_R = \pi - \theta$$
$$= \pi - \frac{3\pi}{4}$$
$$= \frac{\pi}{4}$$

$$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\therefore \sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2} \text{ and } \csc \frac{3\pi}{4} = \frac{2}{\sqrt{2}} \text{ or } \sqrt{2}$$

$$\therefore \csc \left(\frac{-5\pi}{4} \right) = \frac{2}{\sqrt{2}} \text{ or } \sqrt{2}$$

TEXT

Additional Examples
Mathematics 12: p. 213, p. 214 (Example 3)
Mathematics: Principles and Process 12: p. 186 (Examples 1, 2)
Holtmath 12: pp. 289, 291, 292

Exercises

7. If the terminal arm of an angle in standard position lies on the x-axis or y-axis, the angle is called a **quadrantal angle**. Complete the chart by determining the primary trigonometric ratios for the quadrantal angles of a unit circle.

θ	**sin** θ	**cos** θ	**tan** θ
0			
$\frac{\pi}{2}$			
π			
$\frac{3\pi}{2}$			
2π			

8. If $0 \le \theta \le 2\pi$, then determine the values of θ in each of the following:

   a. $\tan \theta$ = undefined

b. $\sin \theta = 1$

c. $\cos \theta = 0$

d. $\tan \theta = 0$

9. Determine the exact value for each of the following:

a. $\sin \frac{10\pi}{3}$

b. $\sec \frac{7\pi}{6}$

c. $\csc \left(-\frac{7\pi}{4} \right)$

d. $\tan 3\pi$

EXERCISE 2c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 215 (Questions 1-3)
Mathematics: Principles and Process 12: p.187 (Questions 6-8)
Holtmath 12: p. 293 (Question 6), p. 295 (Question 8)

Applying the CAST Rule

Watch Video Segment 2d.

Example 1

Find the values of θ for which $\cos\theta = -\frac{\sqrt{3}}{2}$, where $0° \le \theta \le 360°$.

Solution

Use a positive trigonometric ratio to solve for θ_R.

$$\cos\theta_R = \frac{\sqrt{3}}{2}$$

Determine the reference angle. The cosine of 30° is $\frac{\sqrt{3}}{2}$.

$\therefore \theta_R = 30°$

Now determine the two quadrants where the cosine ratio is negative and the measure of the angles in those two quadrants that have a reference angle of 30°.

Cosine is negative in quadrants 2 and 3. The values of θ for which $\cos\theta = -\frac{\sqrt{3}}{2}$ are 150° and 210°.

Example 2

Find the values of θ for which $\sin\theta = \frac{-\sqrt{2}}{2}$, where $0 \le \theta < 2\pi$.

Solution

Solve for θ_R by making the trigonometric ratio positive.

$$\sin\theta_R = \frac{\sqrt{2}}{2}$$

Determine the reference angle. The sin of $\frac{\pi}{4}$ is $\frac{\sqrt{2}}{2}$.

$$\therefore \theta_R = \frac{\pi}{4}$$

Now determine the two quadrants when the sine ratio is negative and the measures of the angles in those two quadrants with a reference angle of $\frac{\pi}{4}$.

Sine is negative in quadrants 3 and 4.

Quadrant 3: $\theta = \pi + \frac{\pi}{4}$

$= \frac{5\pi}{4}$

Quadrant 4: $\theta = \pi + \frac{\pi}{4}$

$= \frac{7\pi}{4}$

The values of θ for which $\sin\theta = \frac{-\sqrt{2}}{2}$ are $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.

Additional Examples
Mathematics 12: p. 205 (Example 3), p. 214 (Example 4)
Mathematics: Principles and Process 12: pp. 176, 177, 232
Holtmath 12: p. 294 (Example 2)

Exercises

10. Solve for θ, where $0° \le \theta \le 360°$.

    a. $\sin\theta = -\frac{1}{2}$

    b. $\sec\theta = -2$

11. Solve for θ, where $0 \le \theta \le 2\pi$.

    a. $\tan \theta = -\sqrt{3}$

    b. $\sin \theta = 0$

12. Solve for θ, where $0 \le \theta \le 4\pi$.

    a. $\tan \theta = -1$

    b. $\sin \theta = \frac{\sqrt{3}}{2}$

Notice that the domain is from 0 to 4π. This means that the terminal arm of θ makes two complete revolutions.

EXERCISE 2d SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 215 (Questions 4-6)
Mathematics: Principles and Process 12: p.187 (Questions 9, 10), p. 232 (Question 1), p. 233 (Questions 2, 3)
Holtmath 12: p. 290 (Questions 7, 8), p. 295 (Questions 9, 10), p. 312 (Question 9), p. 329 (Question 9)

Further Practice – Solving for x, y, r or θ

Challenge yourself

If $\cos\theta = \frac{3}{5}$, where $0° \leq \theta \leq 360°$, then find $\sin\theta$ and $\tan\theta$.

MATH FACTOR

Watch Video Segment 2e.

Example 1

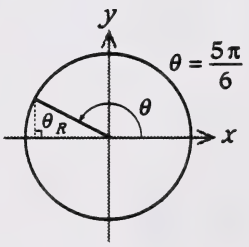
Point $P(x, y)$ is a point on the unit circle as well as a point on the terminal arm of the angle $\frac{5\pi}{6}$. Find the exact values of the coordinates of Point P.

Solution

Find the reference angle, θ_R.

$\frac{5\pi}{6}$ is an angle in quadrant 2.

$$\therefore \theta_R = \pi - \theta$$
$$= \pi - \frac{5\pi}{6}$$
$$= \frac{\pi}{6}$$



Using the reference angle, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sin\frac{\pi}{6} = \frac{1}{2}$.

$\therefore \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ and $\sin\frac{5\pi}{6} = \frac{1}{2}$

So, the exact values of the coordinates of Point $P(x, y)$ are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Example 2

If $\sin\theta = -\frac{5}{13}$ and $\cos\theta < 0$, then determine the exact value of $\tan\theta$.

Solution

$$\sin\theta = -\frac{5}{13}$$

$$\sin\theta = \frac{y}{r}$$

$$= -\frac{5}{13}$$

$\therefore y = -5$ and $r = 13$

Use the Pythagorean Theorem to solve for x.

$$r^2 = x^2 + y^2$$
$$13^2 = x^2 + (-5)^2$$
$$144 = x^2$$
$$\pm 12 = x$$

Use $x = -12$ because $\cos\theta < 0$.

In quadrant 3, $\sin\theta < 0$ and $\cos\theta < 0$.

$\therefore \tan\theta > 0$

$$\tan\theta = \frac{y}{x}$$
$$= \frac{-5}{-12}$$
$$= +\frac{5}{12}$$

TEXT

Additional Examples
Mathematics 12: pp. 205-207
Mathematics: Principles and Process 12: p.178
Holtmath 12: p. 285

Exercises

13. Point $P(x, y)$ is a point on the unit circle as well as a point on the terminal arm of the angle $\frac{7\pi}{4}$. Find the exact values of the coordinates of Point *P*.

14. If $\cos\theta = -\frac{8}{17}$ and $\sin\theta > 0$, then find the exact value of $\cot\theta$.

15. If $\sec\theta = 1.5$, find the measure of θ for the following:

    a. $0° \le \theta < 360°$

    b. $0 \le \theta < 2\pi$

EXERCISE 2e SOLUTIONS are at the back of this workbook.

TEXT

Additional Examples
Mathematics 12: p. 208 (Questions 9-14), p. 266 (Questions 1-13)
Mathematics: Principles and Process 12: p. 174 (Questions 5, 6), p. 175 (Questions 7-11), p. 179 (Questions 2-6, 8), p. 182 (Question 7), p. 189 (Questions 7, 8), p. 190 (Questions 14, 15)
Holtmath 12: p. 283 (Questions 13-16, 18a, 20a), p. 286 (Questions 10-13, 15, 17-19), p. 295 (Question 5), p. 329 (Question 10)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. Using special angle reference triangles is the main building block for **Lesson 3: Applications of Circular Functions**.

Solutions

BUILDING BLOCKS

1. $m^2 = 2^2 - 1^2$ $\quad n^2 = 1^2 + 1^2$
$= \sqrt{3}$ $\quad = \sqrt{2}$

a. $\frac{1}{2}$ $\quad$ b. $\frac{\sqrt{3}}{2}$

c. $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ $\quad$ d. $\frac{\sqrt{3}}{2}$

e. $\frac{1}{2}$ $\quad$ f. $\sqrt{3}$

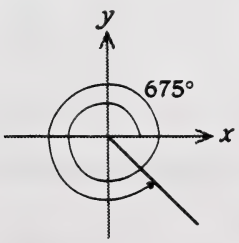
g. $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad$ h. $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

i. 1

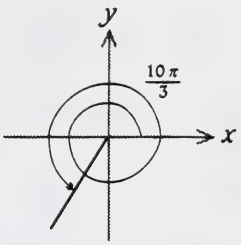
2. a. $\tan\theta = \frac{4}{5}$
$= 0.8$
$\theta \doteq 38.7°$

b. $\sin 32.5° = \frac{14 \text{ cm}}{e}$
$e(\sin 32.5°) = 14 \text{ cm}$
$e = \frac{14 \text{ cm}}{\sin 32.5°}$
$\doteq 26.1 \text{ cm}$

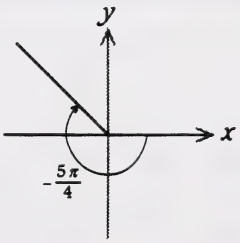
3. a. $675° - 360° = 315°$



b. $\frac{10\pi}{3} - 2\pi = \frac{4\pi}{3}$



c. $-\frac{5\pi}{4} + 2\pi = +\frac{3\pi}{4}$



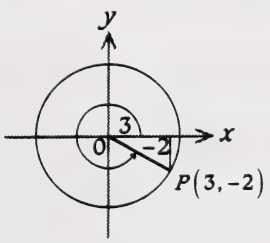
4. a. $\theta_R = 45°$ (the angle from the terminal arm at 315° to the x-axis at 360°)

b. $\theta_R = \frac{\pi}{3}$ (the angle from the terminal arm at $\frac{4\pi}{3}$ to the x-axis at π)

c. $\theta_R = \frac{\pi}{4}$ (the angle from the terminal arm at $\frac{3\pi}{4}$ to the x-axis at π)

EXERCISE 2a

1. a.



b. From the diagram, calculate r.

$$r^2 = x^2 + y^2$$
$$r = \sqrt{(3)^2 + (-2)^2}$$
$$= \sqrt{13}$$

$$\therefore \sin\theta = \frac{-2}{\sqrt{13}} \qquad \csc\theta = \frac{\sqrt{13}}{-2}$$
$$\cos\theta = \frac{3}{\sqrt{13}} \qquad \sec\theta = \frac{\sqrt{13}}{3}$$
$$\tan\theta = \frac{-2}{3} \qquad \cot\theta = \frac{3}{-2}$$

c. To solve for θ_R, use any of the trigonometric ratios in positive form.

$$\tan\theta_R = +\frac{2}{3}$$
$$\therefore \theta_R = 34^\circ$$
So, $\theta = 360^\circ - 34^\circ = 326^\circ$.

d. $1^\circ = \frac{\pi}{180^\circ}$ rad

$$326^\circ = 326^\circ\left(\frac{\pi}{180^\circ}\right) \text{ rad}$$
$$\doteq 5.7 \text{ rad}$$

You can redo question 2c by putting your calculator in radian mode.

e. $\cos\theta = \frac{x}{r}$ and $\sec\theta = \frac{r}{x}$

In quadrant 4, x is positive and the radius is always positive; therefore, cosine and secant will always be positive.

2. $\sin\theta = \frac{y}{r} = \frac{y}{1} = y$ $\qquad \csc\theta = \frac{1}{y}$

$\cos\theta = \frac{x}{r} = \frac{x}{1} = x$ $\qquad \sec\theta = \frac{1}{x}$

$\tan\theta = \frac{y}{x}$ $\qquad \cot\theta = \frac{x}{y}$

3. $P(x, y) = P(\cos\theta, \sin\theta)$ when $P(x, y)$ is the end point on the terminal arm of a circle with a radius of one unit.

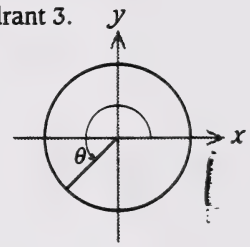
EXERCISE 2b

4.

	a. $\theta_R = \frac{\pi}{6}$	b. $\theta_R = \frac{\pi}{4}$
Quadrant 1	$\theta = \frac{\pi}{6}$	$\theta = \frac{\pi}{4}$
Quadrant 2	$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$	$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$
Quadrant 3	$\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$	$\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$
Quadrant 4	$\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$	$\theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

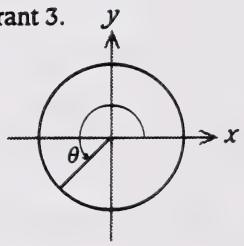
5. a. $\theta = \frac{7\pi}{6}$ is in quadrant 3.

$$\theta_R = \theta - \pi$$
$$= \frac{7\pi}{6} - \pi$$
$$= \frac{\pi}{6}$$



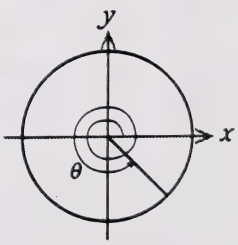
b. $\theta = \frac{5\pi}{4}$ is in quadrant 3.

$$\theta_R = \theta - \pi$$
$$= \frac{5\pi}{4} - \pi$$
$$= \frac{\pi}{4}$$



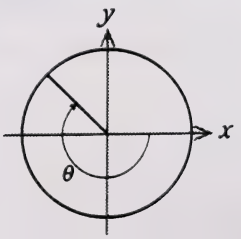
c. $\theta = \frac{11\pi}{3}$ and it is coterminal with $\frac{5\pi}{3}$ in quadrant 4.

$$\theta_R = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$$



d. $\theta = -\frac{5\pi}{4} = +\frac{3\pi}{4}$ and is in quadrant 2.

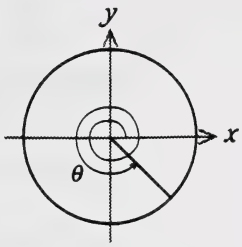
$$\theta_R = \pi - \theta$$
$$= \pi - \frac{3\pi}{4}$$
$$= \frac{\pi}{4}$$



6. a. $\theta_R = \frac{23\pi}{6}$ is in quadrant 4 and is coterminal with $\frac{11\pi}{6}$.

$\theta_R = 2\pi - \frac{11\pi}{6}$

$= \frac{\pi}{6}$



$\sin \frac{\pi}{6} = \frac{1}{2}$

$\sin \frac{23\pi}{6} = -\frac{1}{2}$

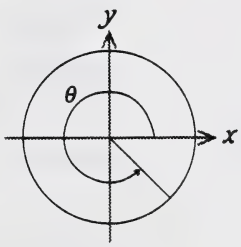
$\sin \frac{23\pi}{6}$ is negative because in quadrant 4, $y < 0$.

b. $\theta = \frac{7\pi}{4}$ is in quadrant 4.

$\theta_R = 2\pi - \theta$

$= 2\pi - \frac{7\pi}{4}$

$= \frac{\pi}{4}$



$\tan \theta = \frac{y}{x}$

$\tan \frac{\pi}{4} = \frac{1}{1} = 1$

$\tan \frac{7\pi}{4} = -1$

$\tan \frac{7\pi}{4}$ is negative because it is in quadrant 4, where x is positive and y is negative.

EXERCISE 2c

7.

θ	$\sin\theta = \frac{y}{r}$	$\cos\theta = \frac{x}{r}$	$\tan\theta = \frac{y}{x}$
0	$\frac{y}{r} = \frac{0}{1} = 0$	$\frac{x}{r} = \frac{1}{1} = 1$	$\frac{y}{x} = \frac{0}{1} = 0$
$\frac{\pi}{2}$	$\frac{y}{r} = \frac{1}{1} = 1$	$\frac{x}{r} = \frac{0}{1} = 0$	$\frac{y}{x} = \frac{1}{0} = \varnothing$
π	$\frac{y}{r} = \frac{0}{-1} = 0$	$\frac{x}{r} = \frac{-1}{1} = -1$	$\frac{y}{x} = \frac{0}{-1} = 0$
$\frac{3\pi}{2}$	$\frac{y}{r} = \frac{-1}{1} = -1$	$\frac{x}{r} = \frac{0}{1} = 0$	$\frac{y}{x} = \frac{-1}{0} = \varnothing$
2π	$\frac{y}{r} = \frac{0}{1} = 0$	$\frac{x}{r} = \frac{1}{1} = 1$	$\frac{y}{x} = \frac{0}{1} = 0$

Note: $\varnothing$ means undefined.

8. a. $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$

b. $\theta = \frac{\pi}{2}$

c. $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$

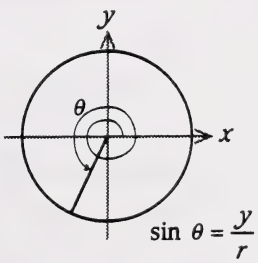
d. $\theta = 0, \pi, 2\pi$

9. a. $\theta = \frac{10\pi}{3}$ is coterminal with $\frac{4\pi}{3}$ in quadrant 3.

$\theta_R = \theta - \pi$

$= \frac{4\pi}{3} - \pi$

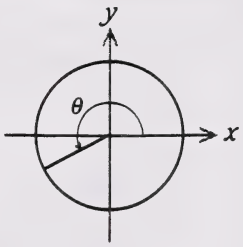
$= \frac{\pi}{3}$



$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\therefore \sin \frac{10\pi}{3} = -\frac{\sqrt{3}}{2}$, since $y < 0$ in quadrant 3.

b. $\theta = \frac{7\pi}{6}$ is in quadrant 3.



$\theta_R = \frac{7\pi}{6} - \pi$

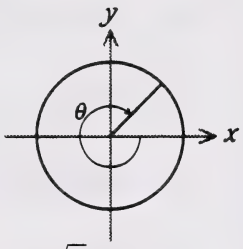
$= \frac{\pi}{6}$

$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\cos \frac{7\pi}{6} = \frac{-\sqrt{3}}{2}$, since $x < 0$ in quadrant 3.

$\therefore \sec \frac{7\pi}{6} = -\frac{2}{\sqrt{3}}$ or $\frac{-2\sqrt{3}}{3}$

c. $\theta = -\frac{7\pi}{4}$ is coterminal with $+\frac{\pi}{4}$ and is in quadrant 1.



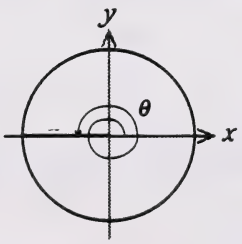
$\theta_R = \frac{\pi}{4}$

$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\therefore \sin\left(\frac{-7\pi}{4}\right) = \frac{\sqrt{2}}{2}$

and $\csc\left(\frac{-7\pi}{4}\right) = \frac{2}{\sqrt{2}}$ or $\sqrt{2}$

d. $\theta = 3\pi$ is coterminal with π.



$\tan \theta = \frac{y}{x}$

$= \frac{0}{-1}$

$\therefore \tan \pi = 0$

$\therefore \tan 3\pi = 0$

EXERCISE 2d

10. a. $\sin \theta_R = \frac{1}{2}$

$\therefore \theta_R = 30°$

$\sin \theta = -\frac{1}{2}$

Quadrant 3: $\theta = 180° + 30°$

$= 210°$

Quadrant 4: $\theta = 360° - 30°$

$= 330°$

$\therefore \theta = 210°$ and $330°$

b. $\sec \theta_R = 2$

$\therefore \cos \theta_R = \frac{1}{2}$

$\theta_R = 60°$

$\cos \theta = -\frac{1}{2}$

Quadrant 2: $\theta = 180° - 60°$

$= 120°$

Quadrant 4: $\theta = 180° + 60°$

$= 240°$

$\therefore \theta = 120°$ and $240°$

11. a. $\tan \theta_R = \sqrt{3}$

$\theta_R = \frac{\pi}{3}$

$\tan \theta = -\sqrt{3}$

Quadrant 2: $\theta = \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$

Quadrant 4: $\theta = 2\pi - \frac{\pi}{3}$

$= \frac{5\pi}{3}$

$\therefore \theta = \frac{2\pi}{3}$ and $\frac{5\pi}{3}$

b. $\sin \theta = 0$

$\therefore \theta = 0, \pi, 2\pi$

12. a. $\tan \theta_R = 1$

$\theta_R = \frac{\pi}{4}$

$\tan \theta = -1$

Quadrant 2: $\theta = \pi - \frac{\pi}{4}$

$= \frac{3\pi}{4}$

Second revolution: $\theta = \frac{3\pi}{4} + 2\pi$

$= \frac{11\pi}{4}$

Quadrant 4: $\theta = 2\pi - \frac{\pi}{4}$

$= \frac{7\pi}{4}$

Second revolution: $\theta = \frac{7\pi}{4} + 2\pi$

$= \frac{15\pi}{4}$

$\therefore \theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$ and $\frac{15\pi}{4}$

b. $\sin \theta_R = \frac{\sqrt{3}}{2}$

$\theta_R = \frac{\pi}{3}$

$\sin \theta = \frac{\sqrt{3}}{2}$

Quadrant 1: $\theta = \frac{\pi}{3}$

Second revolution: $\theta = \frac{\pi}{3} + 2\pi$

$= \frac{7\pi}{3}$

Quadrant 2: $\theta = \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$

Second revolution: $\theta = \frac{2\pi}{3} + 2\pi$

$= \frac{8\pi}{3}$

$\therefore \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}$ and $\frac{8\pi}{3}$

EXERCISE 2e

13. $\theta = \frac{7\pi}{4}$ $\quad \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\quad \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\theta_R = \frac{\pi}{4}$ $\quad \cos \frac{7\pi}{4} = +\frac{\sqrt{2}}{2}$ $\quad \sin \frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$

$\therefore x = \frac{\sqrt{2}}{2}$ $\quad \therefore y = -\frac{\sqrt{2}}{2}$

$\therefore P(x, y) = P\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

14. Cosine is negative and sine is positive in quadrant 2.

$\cos \theta = \frac{x}{r}$

$= \frac{-8}{17}$

$x = -8$ and $r = 17$

$\cot \theta = \frac{x}{y}$

$= \frac{-8}{15}$

$r^2 = x^2 + y^2$

$(17)^2 = (-8)^2 + y^2$

$289 = 64 + y^2$

$y^2 = 255$

$\therefore y = +15$ in quadrant 2

15. $\sec \theta_R = +1.5$

$\cos \theta_R = \frac{1}{1.5}$

$\doteq 0.6$

a. $\theta_R \doteq 48^\circ$

Quadrant 2: $\theta = 180^\circ - 48^\circ$

$= 132^\circ$

Quadrant 3: $\theta = 180^\circ + 48^\circ$

$= 228^\circ$

$\therefore \theta = 132^\circ$ and 228°

b. $\theta_R \doteq 48^\circ \left(\frac{\pi}{180^\circ}\right)$ rad

$\doteq 0.84$ rad

Quadrant 2: $\theta = \pi - 0.84$

$\doteq 2.3$ rad

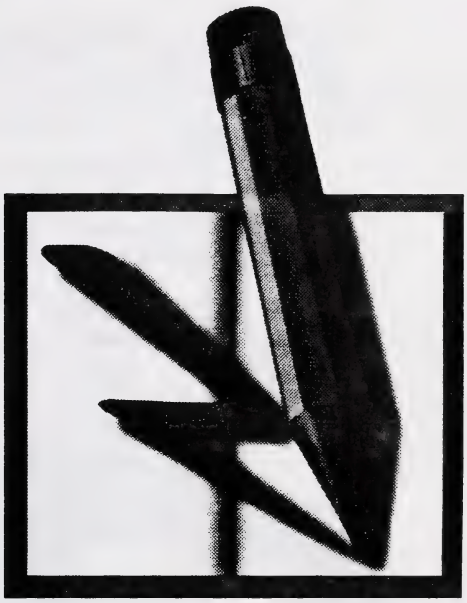
Quadrant 3: $\theta = \pi + 0.84$

$\doteq 3.98$ rad

$\therefore \theta \doteq 2.3$ and 3.98 rad

MATH FACTOR

TRIGONOMETRIC AND CIRCULAR FUNCTIONS



Lesson 3:
Applications of Circular Functions

Student Workbook

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This guide accompanies **Applications of Circular Functions** from the Math Factor series.

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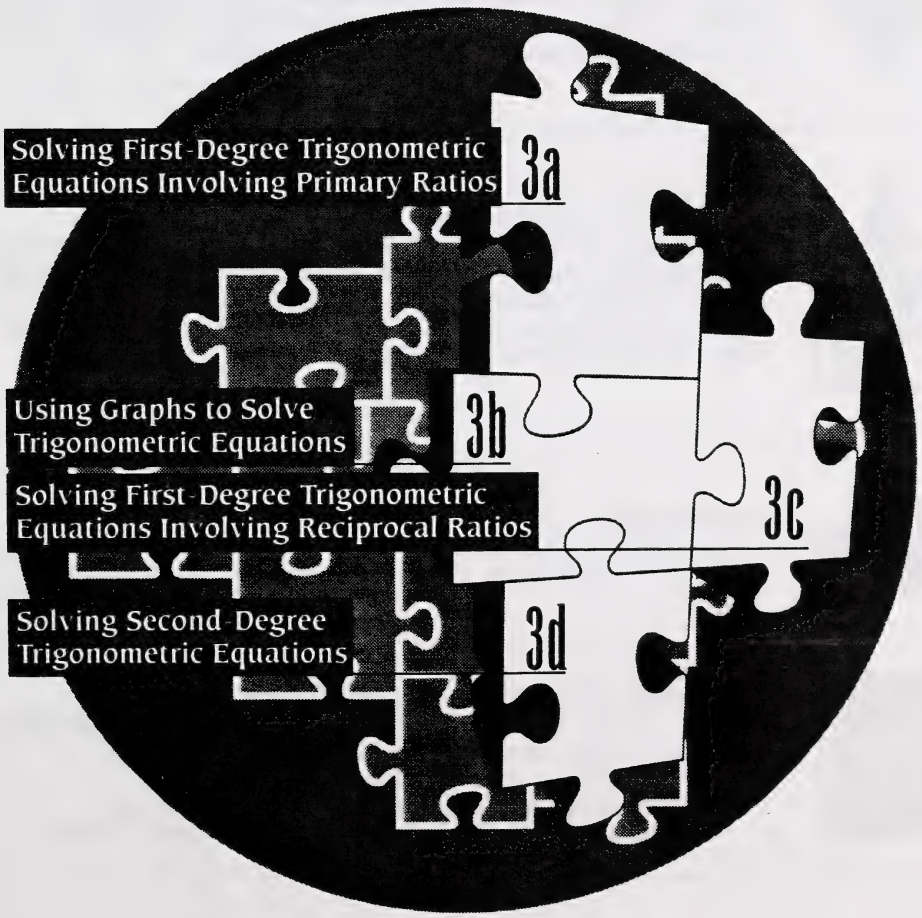
Lesson 3: APPLICATIONS OF CIRCULAR FUNCTIONS

LEARNER EXPECTATIONS

By the end of Lesson 3 you should be able to:

- solve for all possible solutions, θ, within a given domain for first-degree trigonometric equations
- solve for all possible solutions, θ, within a given domain for second-degree trigonometric equations
- use graphs to solve trigonometric equations
- explain the relationship between the roots of a trigonometric equation and the graph of the corresponding function

This lesson has four parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

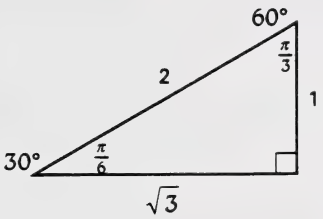
This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

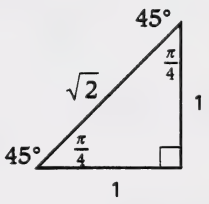
- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Building Blocks

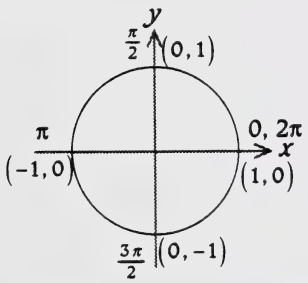
In this lesson, you will solve trigonometric equations where the angle, θ, is unknown. The special angle triangles and quadrantal angles are frequently used when solving for θ.

Special Angles

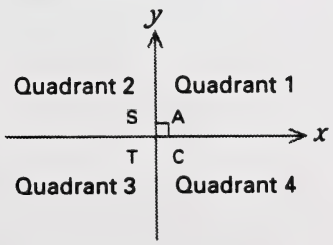




Quadrantal Angles



Cast Rule



Example 1

If $\sin\theta = \frac{-\sqrt{3}}{2}$, find θ, where $0 \le \theta \le 2\pi$.

Solution

$\theta_R = \frac{\pi}{3}$

Quadrant 3: $\theta = \pi + \frac{\pi}{3}$

$= \frac{4\pi}{3}$

Quadrant 4: $\theta = 2\pi - \frac{\pi}{3}$

$= \frac{5\pi}{3}$

$\therefore \theta = \frac{4\pi}{3}$ and $\frac{5\pi}{3}$

Example 2

If $\sin\theta = 0$, find θ, where $0 \leq \theta \leq 2\pi$.

Solution

$\therefore \theta = 0, \pi$ and 2π

Example 3

If $\tan\theta = 1.327$, find θ, where $0 \leq \theta \leq 2\pi$.

Solution

You can solve for θ in degrees and then convert the degrees to radians by multiplying by $\frac{\pi}{180°}$; or you can put your calculator in radian mode.

$\theta_R \doteq 53° \doteq 0.93$ rad

Quadrant 1: $\theta \doteq 0.93$ rad

Quadrant 3: $\theta \doteq \pi + 0.93$

$\doteq 3.14 + 0.93$

$\doteq 4.07$ rad

$\therefore \theta \doteq 0.93$ rad and 4.07 rad

Exercises

1. Solve for θ.

   a. $\tan \theta = \pm\frac{\sqrt{3}}{3}$, where $0^\circ \leq \theta \leq 360^\circ$

   b. $\cos \theta = -0.2588$, where $0 \leq \theta \leq 2\pi$

BUILDING BLOCKS SOLUTIONS are at the back of this workbook.

Solving First-Degree Trigonometric Equations Involving Primary Ratios

Solve the following trigonometric equations.

a. $2\sin\theta = \sqrt{3},\ 0 \le \theta \le 2\pi$ b. $\cos\theta - 1 = 0,\ 0^\circ \le \theta \le 360^\circ$

Watch the Introduction and Video Segment 3a.

Example 1

Solve the equation $2\sin\theta - 1 = 0$, where $0^\circ \le \theta < 360^\circ$.

Solution

Notice that the domain indicates that the value of the angle must be between 0° and 360°.

Isolate the trigonometric function by using your skills of solving algebraic equations.

$$2\sin\theta - 1 = 0$$
$$2\sin\theta = 1$$
$$\sin\theta = \frac{1}{2}$$
$$\theta_R = 30^\circ$$

Quadrant 1: $\theta = 30^\circ$

Quadrant 2: $\theta = 180^\circ - 30^\circ$
$= 150^\circ$

$\therefore \theta = 30^\circ$ and 150°

Example 2

Solve the equation $4\cos\theta + 1 = -2$, where $0 \le \theta < 2\pi$.

Solution

Notice that the domain is in radians, so the values of the angles must be in radians and between 0 and 2π.

$$4\cos\theta + 1 = -2$$
$$4\cos\theta = -2 - 1$$
$$4\cos\theta = -3$$
$$\cos\theta = \frac{-3}{4}$$
$$\cos\theta_R = \frac{+3}{4}$$
$$\theta_R \doteq 0.72 \text{ rad}$$

Quadrant 2: $\theta \doteq \pi - 0.72$ rad
$\doteq 3.14 - 0.72$ rad
$\doteq 2.42$ rad

Quadrant 3: $\theta \doteq \pi + 0.72$ rad
$\doteq 3.14 + 0.72$ rad
$\doteq 3.86$ rad

$\therefore \theta \doteq 2.42$ rad and 3.86 rad

Additional Examples
Mathematics 12: pp. 205-207, p. 214 (Example 4)
Holtmath 12: p. 294 (Example 2)

Exercises

1. Solve each of the following equations.

   a. $\sin\theta - 1 = 3\sin\theta$, where $0 \leq \theta < 2\pi$.

   b. $2\cos\theta + \sqrt{2} = 0$, where $0 \leq \theta < 2\pi$.

   c. $4\tan\theta - 1 = \tan\theta$, where $0 \leq \theta < 2\pi$.

   d. $5\sin\theta + 4 = 0$, where $0 \leq \theta < 2\pi$.

EXERCISE 3a SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 208 (Questions 5, 9, 10), p. 215 (Questions 4-6)
Mathematics: Principles and Process 12: p. 187 (Question 10), p. 232 (Question 1), p. 233 (Questions 2, 3, 7a, 7b)
Holtmath 12: p. 290 (Questions 7, 8)

Using Graphs to Solve Trigonometric Equations

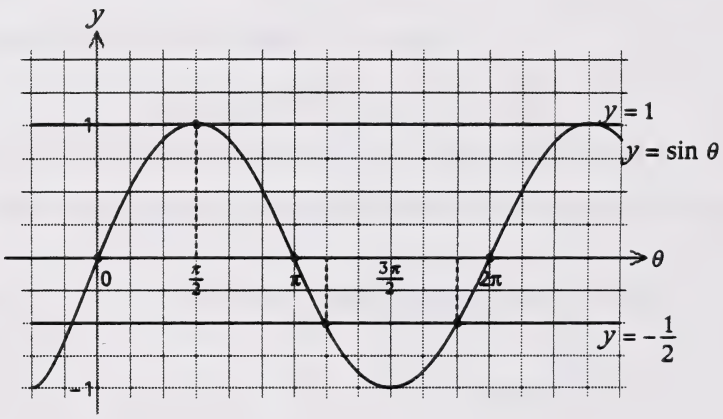
MATH FACTOR

Watch Video Segment 3b.

Example

Given the graph $y = \sin\theta$, find the solutions to each of the following trigonometric equations.

a. $\sin\theta = 0, 0 \le \theta \le 2\pi$ b. $\sin\theta = 1, 0 \le \theta \le 2\pi$ c. $\sin\theta = -\frac{1}{2}, 0 \le \theta \le 2\pi$



Solution

a. $\sin\theta = 0$
$\sin\theta = 0$, when $y = 0$
$\theta = 0, \pi, 2\pi$ for $0 \le \theta \le 2\pi$

b. $\sin\theta = 1$
To solve $\sin\theta = 1$, draw the horizontal line $y = 1$. The θ-coordinates of the points of intersection for the graphs $y = 1$ and $y = \sin\theta$ are the solutions.
$\theta = \frac{\pi}{2}$ for $0 \le \theta \le 2\pi$

c. $\sin\theta = -\frac{1}{2}$

To solve for $\sin\theta = -\frac{1}{2}$, draw the horizontal line $y = -\frac{1}{2}$. The θ-coordinates of the points of intersection for the graphs $y = -\frac{1}{2}$ and $y = \sin\theta$ are the solutions.

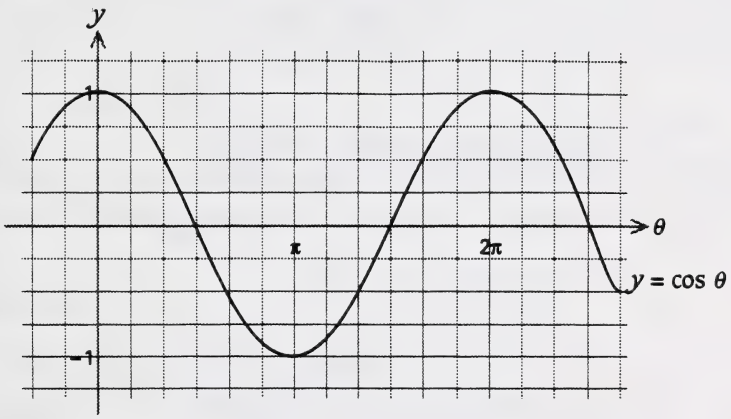
As shown, $\sin\theta = -\frac{1}{2}$ when $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ for $0 \le \theta \le 2\pi$.

TEXT

Additional Examples
Mathematics: Principles and Process 12: p. 230
Holtmath 12: p. 327 (Example 1)

Exercises

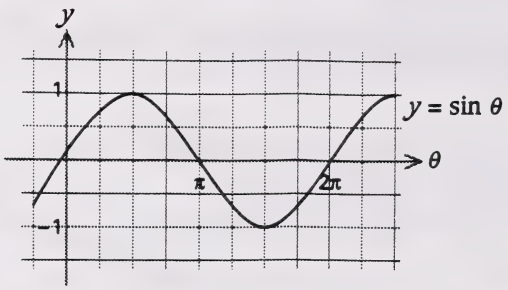
2. Use the graph of $y = \cos\theta$ to solve the equations.



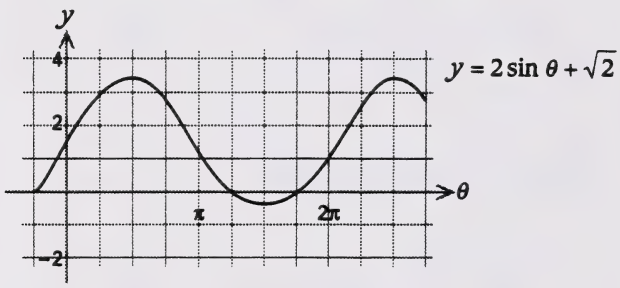
a. $\cos\theta = \frac{\sqrt{3}}{2}$, where $0 \le \theta \le 2\pi$

b. $\cos\theta = 0$, where $0 \le \theta \le 2\pi$

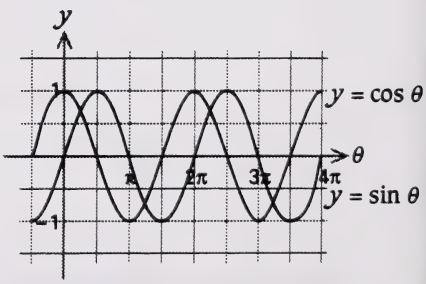
3. a. Use the graph of $y = \sin\theta$ to solve $\sin\theta = -\frac{\sqrt{2}}{2}$, where $0 \le \theta \le 2\pi$.



b. The graph of $y = 2\sin\theta + \sqrt{2}$ is shown. Explain how this graph can be used to solve the equation $\sin\theta = \frac{-\sqrt{2}}{2}$.



4. a. The graphs of $y = \cos\theta$ and $y = \sin\theta$ are shown to the right. Explain how you can use these graphs to approximate the solution to the equation $\cos\theta = \sin\theta$, where $0 \le \theta \le 4\pi$.



b. Solve the equation $\cos\theta = \sin\theta$, where $0 \le \theta \le 4\pi$.

EXERCISE 3b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics: Principles and Process 12: p. 230 (Question 1), p. 231 (Question 2)
Holtmath 12: p. 329 (Question 8)

Solving First-Degree Trigonometric Equations Involving Reciprocal Ratios

Solve these trigonometric equations, where $0 \le \theta < 2\pi$.

a. $\csc \theta = \sqrt{2}$ b. $3\sec^2 \theta = 4$

$\sec^2 \theta = (\sec \theta)^2$

Watch Video Segment 3c.

Example 1

Solve the equation $\sec \theta - 2 = 0$, where $0 \le \theta \le 6\pi$.

Solution

$\sec \theta = 2$

$\cos \theta = \frac{1}{2}$

$\theta_R = \frac{\pi}{3}$

Revolution 1: $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Revolution 2: $\theta = \frac{\pi}{3} + 2\pi, \frac{5\pi}{3} + 2\pi$ or $\frac{7\pi}{3}, \frac{11\pi}{3}$

Revolution 3: $\theta = \frac{7\pi}{3} + 2\pi, \frac{11\pi}{3} + 2\pi$ or $\frac{13\pi}{3}, \frac{17\pi}{3}$

$\therefore \theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}$

Example 2

Solve the equation $\sqrt{3}\cot \theta = 1$, where $0 \le \theta \le 2\pi$.

Solution

$\cot \theta = \frac{1}{\sqrt{3}}$

$\tan \theta = \sqrt{3}$

$\theta_R = \frac{\pi}{3}$

$\theta = \frac{\pi}{3}, \frac{4\pi}{3}$

Exercises

5. Solve the following equations.

   a. $\sqrt{3}\csc\theta - 2 = 0$, where $0° \le \theta \le 720°$

   b. $3\csc^2\theta = 6$, where $0 \le \theta \le 2\pi$

   c. $\cot\theta = -\sqrt{3}$, where $0 \le \theta \le \pi$

EXERCISE 3c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 215 (Questions 4-6)
Mathematics: Principles and Process 12: p. 187 (Questions 10c, 10d), p. 233 (Questions 8a, 8b)
Holtmath 12: p. 295 (Questions 9, 10), p. 312 (Question 9), p. 329 (Questions 9-11, 13)

Solving Second-Degree Trigonometric Equations

Challenge yourself

Solve the equation $2\sin^2\theta - 5\sin\theta - 3 = 0$, where $0° \le \theta \le 360°$.

MATH FACTOR

Watch Video Segment 3d.

Example 1

Solve the equation $\sin^2\theta = \sin\theta$, where $0 \le \theta \le 2\pi$.

Solution

$\sin^2\theta - \sin\theta = \theta$

$\sin\theta(\sin\theta - 1) = 0$

$\therefore \sin\theta = 0$ or $\sin\theta - 1 = 0$

$\sin\theta = +1$

$\theta = 0, \frac{\pi}{2}, \pi, 2\pi$

Example 2

Solve the equation $2\csc^2\theta - 2 = 3\csc\theta$, where $0 \le \theta \le 2\pi$.

Solution

$2\csc^2\theta - 3\csc\theta - 2 = 0$

$(2\csc\theta + 1)(\csc\theta - 2) = 0$

$\therefore 2\csc\theta + 1 = 0$ or $\csc\theta - 2 = 0$

$\csc\theta = -\frac{1}{2}$ $\qquad$ $\csc\theta = 2$

$\sin\theta = -2$ $\qquad$ $\sin\theta = \frac{1}{2}$

θ has no solution. $\qquad$ $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

The trigonometric values for $\sin\theta$ and $\cos\theta$ have maximum values of $+1$ and minimum values of -1.

$-1 \le \sin\theta \le 1$

$-1 \le \cos\theta \le 1$

Additional Examples
Mathematics 12: p. 289 (Example 1), p. 290 (Example 2)
Mathematics: Principles and Process 12: p. 232

Exercises

6. Solve the following equations.

   a. $\cot^2 \theta = \cot \theta$, where $0 \le \theta \le 2\pi$

   b. $4\cos^2 \theta + \cos \theta = 3$, where $0 \le \theta \le 2\pi$

   c. $4\cos^2 \theta - 2\cos \theta - 2\sqrt{3}\cos \theta + \sqrt{3} = 0$, where $0 \le \theta \le 2\pi$

   d. $\sin \theta \cos \theta - \cos \theta + \sin \theta - 1 = 0$, where $0 \le \theta \le 2\pi$

EXERCISE 3d SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 293 (Questions 1, 2, 3a, 3b)
Mathematics: Principles and Process 12: p. 233 (Questions 5, 6, 7a, 7c, 7d, 8)
Holtmath 12: p. 329 (Questions 13, 14)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the ones that you have mastered. You will learn more about the graphs of trigonometric functions in **Lesson 4: Graphing Trigonometric Functions**.

Solutions

BUILDING BLOCKS

1. a. $\theta_R = 30°$

$\theta = 30°, 150°, 210°$ and $330°$

Since $\tan\theta = \frac{+\sqrt{3}}{3}$ and $\frac{-\sqrt{3}}{3}$, there are four solutions. The domain is in degrees, so the answer must be in degrees.

b. $\cos\theta = -0.2588$

$\cos\theta_R = +0.2588$

$\theta_R \doteq 1.31$ rad

The reference angle is found by entering a positive ratio into the calculator. Notice that the domain is in radians.

Quadrant 2: $\theta \doteq 3.14 - 1.31$

$\theta \doteq 1.83$ rad

Quadrant 3: $\theta \doteq 3.14 + 1.31$

$\theta \doteq 4.45$ rad

$\therefore \theta \doteq 1.83$ rad and 4.45 rad

EXERCISE 3a

1. a. $\sin\theta - 1 = 3\sin\theta$

$-2\sin\theta = 1$

$\sin\theta = -\frac{1}{2}$

$\theta_R = \frac{\pi}{6}$

$\therefore \theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$

b. $2\cos\theta + \sqrt{2} = 0$

$\cos\theta = \frac{-\sqrt{2}}{2}$

$\theta_R = \frac{\pi}{4}$

$\therefore \theta = \frac{3\pi}{4}$ and $\frac{5\pi}{4}$

c. $4\tan\theta - 1 = \tan\theta$

$3\tan\theta = 1$

$\tan\theta = \frac{1}{3}$

$\theta_R \doteq 0.32$ rad

$\theta \doteq 0.32$ rad and 3.46 rad

d. $\sin\theta = \frac{-4}{5}$

$\theta_R \doteq 0.93$ rad

$\theta \doteq 4.07$ rad and 5.35 rad

EXERCISE 3b

2. a. The solutions of $\cos\theta = \frac{\sqrt{3}}{2}$, where $0 \le \theta \le 2\pi$, are the θ-coordinates of the points of intersection of the two graphs $y = \frac{\sqrt{3}}{2}$ and $y = \cos\theta$.

$\theta = \frac{\pi}{6}, \frac{11\pi}{6}$

b. The solutions of $\cos\theta = 0$, where $0 \le \theta \le 2\pi$, are the θ-coordinates of the points of intersection of the two graphs $y = 0$ and $y = \cos\theta$. Recall that at $y = 0$, you are finding the θ-intercepts for $y = \cos\theta$.

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

3. a. The solutions are the θ-coordinates of the points of intersection of the two graphs $y = \sin\theta$ and $y = \frac{-\sqrt{2}}{2}$.

$\theta = \frac{5\pi}{4}, \frac{7\pi}{4}$

b. The θ-intercepts of the graph of $y = 2\sin\theta + \sqrt{2}$ are the same as the solutions to the equation $\sin\theta = \frac{-\sqrt{2}}{2}$.

$\sin\theta = \frac{-\sqrt{2}}{2}$ can be rewritten as $2\sin\theta + \sqrt{2} = 0$.

4. a. The solutions to the equation are the points of intersection of the two graphs; therefore, they can be approximated as $\frac{\pi}{4}$, $\frac{5\pi}{4}$, $\frac{9\pi}{4}$ and $\frac{13\pi}{4}$.

b. $\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

EXERCISE 3c

5. a. $\csc\theta = \frac{2}{\sqrt{3}}$, $\sin\theta = \frac{\sqrt{3}}{2}$

$\theta_R = 60°$

$\therefore \theta = 60°, 120°, 420°$ and $480°$

For the domain $0° \le \theta \le 720°$, you do two revolutions producing four solutions.

b. $\csc^2\theta = 2$

$\csc\theta = \pm 2$

$\sin\theta = \pm\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$

$\theta_R = \frac{\pi}{4}$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

c. $\cot\theta = -\sqrt{3}$, $0 \le \theta \le \pi$

$\tan\theta = \frac{-1}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$

$\theta_R = \frac{\pi}{6}$

$\therefore \theta = \frac{5\pi}{6}$

EXERCISE 3d

6. a. $\cot^2\theta - \cot\theta = 0$

$\cot\theta(\cot\theta - 1) = 0$

$\therefore \cot\theta = 0$ or $\cot\theta - 1 = 0$

$\tan\theta$ is undefined. $\cot\theta = 1$

$\therefore \theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\tan\theta = 1$

$\theta_R = \frac{\pi}{4}$

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$\therefore \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{5\pi}{4}$

b. $4\cos^2\theta + \cos\theta - 3 = 0$

$(\cos\theta + 1)(4\cos\theta - 3) = 0$

$\therefore \cos\theta + 1 = 0$ or $4\cos\theta - 3 = 0$

$\cos\theta = -1$ $\cos\theta = \frac{3}{4}$

$\therefore \theta = \pi$

$\theta_R \doteq 0.72$ rad

$\theta \doteq 0.72$ rad, 5.56 ra

$\therefore \theta \doteq 0.72, 5.56, \pi$

c. $\left(4\cos^2\theta - 2\cos\theta\right) + \left(-2\sqrt{3}\cos\theta + \sqrt{3}\right) = 0$

$2\cos\theta(2\cos\theta - 1) - \sqrt{3}(2\cos\theta - 1) = 0$

$(2\cos\theta - 1)\left(2\cos\theta - \sqrt{3}\right) = 0$

$\therefore 2\cos\theta - 1 = 0$ or $2\cos\theta - \sqrt{3} = 0$

$\cos\theta = \frac{1}{2}$ $\cos\theta = \frac{\sqrt{3}}{2}$

$\theta_R = \frac{\pi}{3}$ $\theta_R = \frac{\pi}{6}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ $\theta = \frac{\pi}{6}, \frac{11\pi}{6}$

$\therefore \theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}$

d. $(\sin\theta\cos\theta - \cos\theta) + (\sin\theta - 1) = 0$

$\cos\theta(\sin\theta - 1) + 1(\sin\theta - 1) = 0$

$(\sin\theta - 1)(\cos\theta + 1) = 0$

$\therefore \sin\theta - 1 = 0$ or $\cos\theta + 1 = 0$

$\sin\theta = 1$ $\cos\theta = -1$

$\theta = \frac{\pi}{2}$ $\theta = \pi$

$\therefore \theta = \frac{\pi}{2}, \pi$

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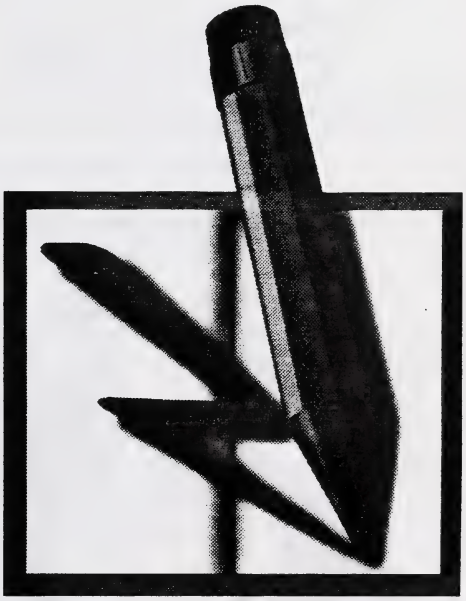
MATH FACTOR

TRIGONOMETRIC AND CIRCULAR FUNCTIONS



Lesson 4:
Graphing Trigonometric Functions

Student Workbook

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This guide accompanies **Graphing Trigonometric Functions** from the Math Factor series.

Writer: Shauna Boyce
Editor: Wilma Gonsalves
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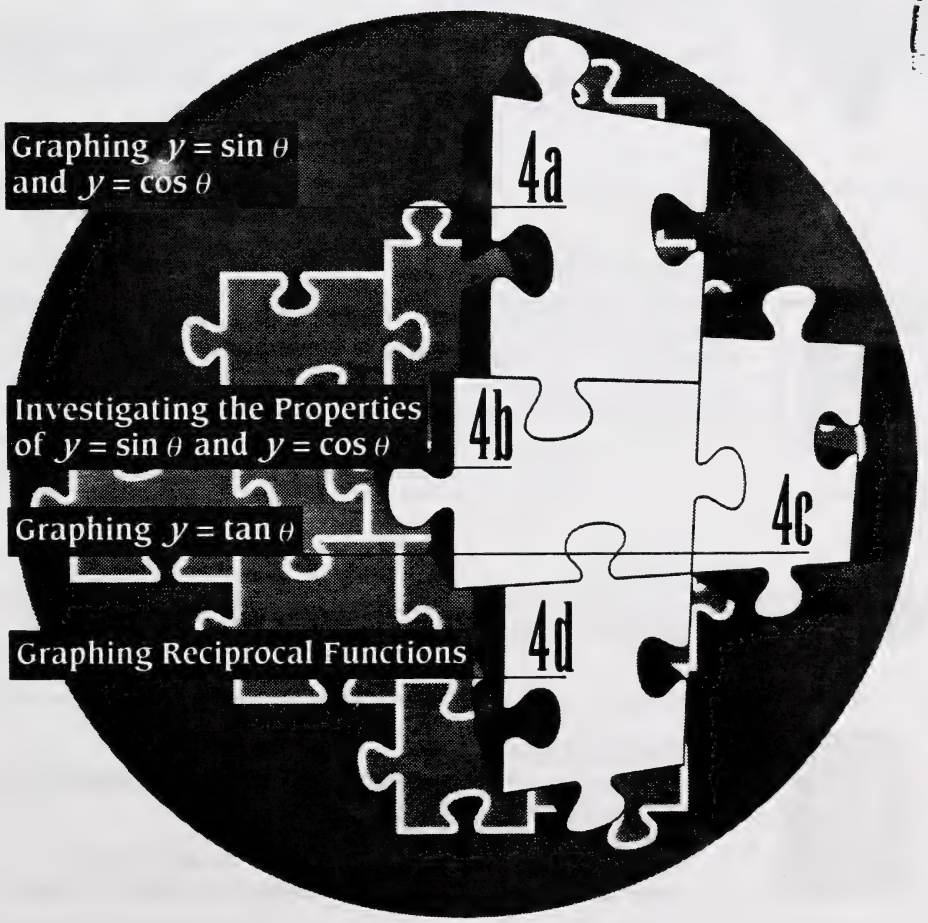
Lesson 4: GRAPHING TRIGONOMETRIC FUNCTIONS

LEARNER EXPECTATIONS

By the end of Lesson 4 you should be able to:

- understand the relationship between the graphs of trigonometric functions and circular functions
- sketch graphs of the trigonometric functions sin θ, cos θ, and tan θ
- compare and contrast their properties
- define and use period, amplitude and asymptotes

This lesson has four parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

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- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Graphing $y = \sin\theta$ and $y = \cos\theta$

MATH FACTOR

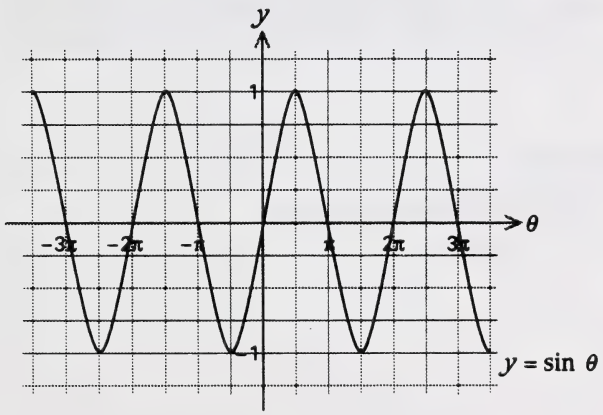
Watch the Introduction and Video Segment 4a.

Example 1

Use the following table of values to sketch the graph of $y = \sin\theta$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	$\frac{5\pi}{3}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

Solution

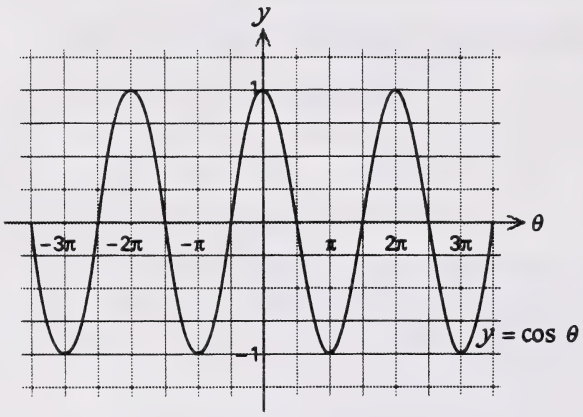


Example 2

Use the table of values to sketch the graph of $y = \cos\theta$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

Solution



TEXT

Additional Examples
Mathematics 12: pp. 216-218
Mathematics: Principles and Process 12: p. 191
Holtmath 12: p. 296 (Example 1)

Exercises

1. Refer to the graphs in Examples 1 and 2 to approximate the values of each of the following:

   a. $\sin \frac{\pi}{4} =$ _____

   b. $\cos \frac{5\pi}{4} =$ _____

   c. $\cos\left(-\frac{\pi}{4}\right) =$ _____

   d. $\sin\left(-\frac{\pi}{2}\right) =$ _____

EXERCISE 4a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: pp. 219-220 (Questions 1, 3)
Mathematics: Principles and Process 12: p. 192 (Questions 1-3)
Holtmath 12: p. 298 (Questions 1, 5, 7)

Investigating the Properties of $y = \sin\theta$ and $y = \cos\theta$

Challenge yourself

Describe the similarities and differences between the graphs of $y = \sin\theta$ and $y = \cos\theta$.

MATH FACTOR

Watch Video Segment 4b.

In order to draw a quick sketch of $y = \sin\theta$ or $y = \cos\theta$, you should know the following:

- θ-intercepts
- maximum value
- minimum value
- y-intercept

Example 1

Sketch the graph of $y = \sin\theta$.

Solution

The θ-intercepts are $-3\pi,\ -2\pi,\ -\pi,\ 0,\ \pi,\ 2\pi,\ 3\pi,\ \ldots$ (or $n\pi,\ n \in I$).

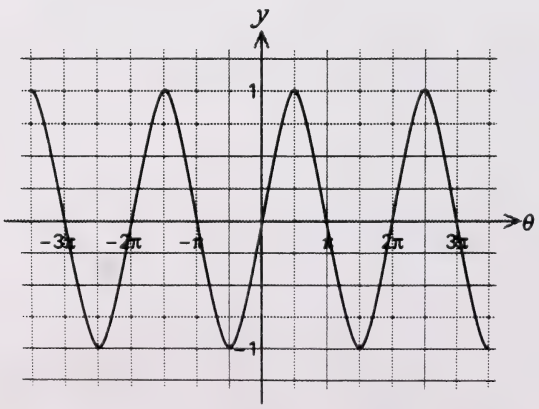
The maximum value is 1.

The minimum value is -1.

Maximum and minimum values occur halfway between the θ-intercepts.

The y-intercept is 0.

After you plot these points, you can join them with a smooth curve. By continuing the pattern to the left or right, you can extend the graph.



Example 2

Sketch the graph of $y = \cos\theta$.

Solution

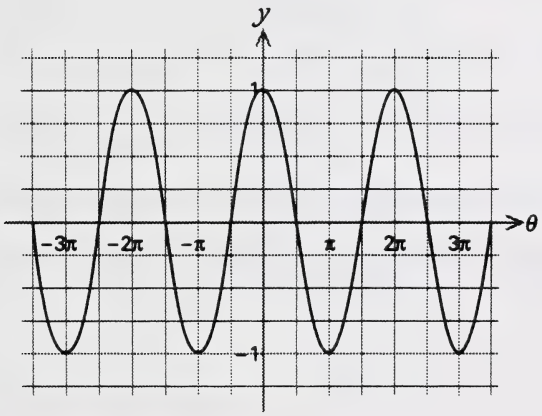
The θ-intercepts are $-\frac{3\pi}{2}$, $-\frac{\pi}{2}$, $\frac{\pi}{2}$, $\frac{3\pi}{2}$, ... $\left(n\pi + \frac{\pi}{2}, n \in \mathrm{I}\right)$.

The maximum value is 1.

The minimum value is -1.

The y-intercept is 1.

Use the method described in Example 1 to sketch the graph of $y = \sin\theta$.



TEXT

Additional Examples
Mathematics 12: p. 218 (Notes in box)
Mathematics: Principles and Process 12: p. 192 (I-IV)
Holtmath 12: p. 296 (Example 1)

Exercises

2. Complete the following chart for $y = \sin\ \theta$ and $y = \cos\ \theta$.

Property	$y = \sin\ \theta$	$y = \cos\ \theta$
amplitude		
maximum/minimum		
period		
y-intercepts		
θ-intercepts		

3. Describe a sinusoidal curve.

4. Give three examples of sinusoidal curves found in nature or science.

EXERCISE 4b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 220 (Questions 4-8)
Mathematics: Principles and Process 12: p. 192 (Question 4), p. 193 (Question 6)
Holtmath 12: p. 298 (Questions 4, 6a, 6b, 6c, 6d)

Graphing $y = \tan\theta$

challenge yourself

Compare and contrast the graphs of the three primary trigonometric functions.

MATH FACTOR

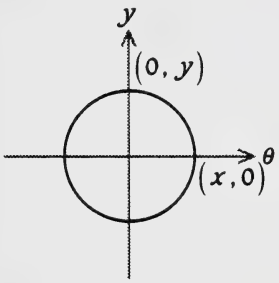
Watch Video Segment 4c.

For angles close to $\frac{\pi}{2}\left(90^\circ\right)$ or $\frac{3\pi}{2}\left(270^\circ\right)$, the values of $\tan\theta$ are either very large and positive or very small and negative.

For example, $\tan 89.9^\circ = 572.95$ and $\tan 90.1^\circ = -572.95$.

Using a calculator to try to find $\tan 90^\circ$ should prompt an error message because this value is undefined.

Using the definition $\tan\theta = \frac{y}{x}$, you can see from the diagram that $\tan 90^\circ = \frac{y}{0}$.



$$\tan\theta = \frac{y}{x} = \frac{y}{0}$$

On a graph, this is represented by a dotted, vertical line called an **asymptote**. An asymptote is not really part of the graph, and some graphing programs will not display it. It is a symbolic way of showing that at that value of θ, the function $\tan\theta$ is undefined; therefore, it is excluded from the domain of $y = \tan\theta$.

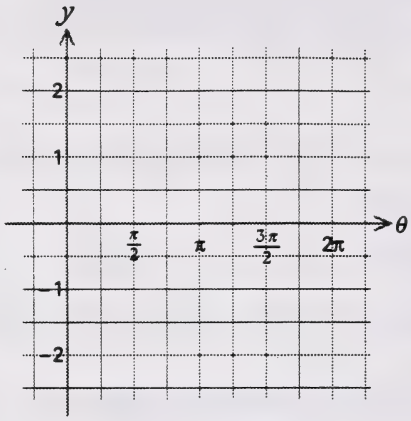
TEXT

Additional Examples
Mathematics 12: p. 222, p. 223 (Notes in box)
Holtmath 12: p. 297 (Example 2)

Exercises

5. Complete the table of values for $y = \tan\theta$ and sketch the graph.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\tan\theta$													



6. Complete the following chart for the graph of $y = \tan\theta$.

Property	$y = \tan\theta$
amplitude	
maximum/minimum	
period	
y-intercepts	
θ-intercepts	

7. State whether each of the following are true (T) or false (F).

_____ a. The range of the sine function is $-1 \leq \sin\theta \leq 1$.

_____ b. The domain of the tangent function is $\{\theta \mid \theta \in R\}$.

_____ c. The amplitude of the graph of the cosine function is 1.

_____ d. The zeros of the sine function and tangent function are the same.

_____ e. The range of the tangent function is $\tan\theta \in R$.

_____ f. The amplitude of the graph of the tangent function is 1.

_____ g. The tangent function is undefined at the zeros of the cosine function.

EXERCISE 4c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 224 (Questions 1-4)
Mathematics: Principles and Process 12: p. 193 (Questions 7, 8)
Holtmath 12: p. 298 (Questions 2, 6e, 6f, 6g, 6h)

Graphing Reciprocal Functions

MATH FACTOR

Watch Video Segment 4d.

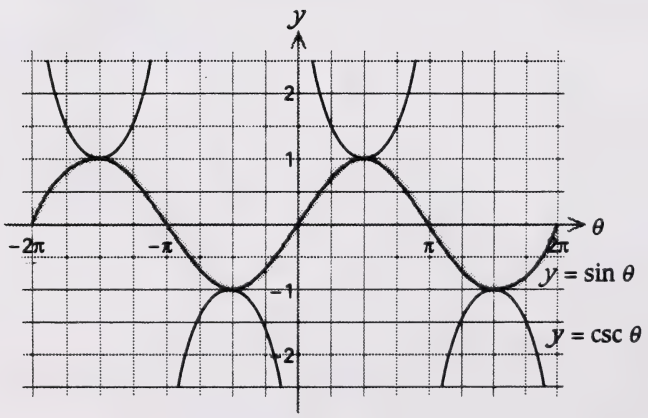
Example

Sketch the graph of $y = \csc\theta$.

Solution

Here are some hints for sketching $y = \csc\theta$.

- Sketch $y = \sin\theta$.
- When $\sin\theta = 0$, $\csc\theta = \frac{1}{0}$ which is undefined. $\sin\theta = 0$ at $n\pi$. Therefore, there are asymptotes at $\theta = n\pi$.
- When $\sin\theta = 1$, $\csc\theta = 1$; when $\sin\theta = -1$, $\csc\theta = -1$. Plot these points.
- Complete the graph of $y = \csc\theta$ as the graph shown.

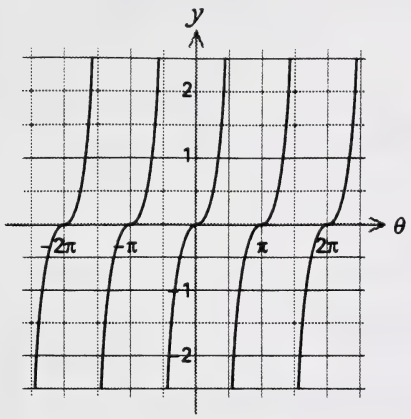


TEXT

Additional Examples
Mathematics: Principles and Process 12: pp. 206, 207
Holtmath 12: p. 297 (Example 3)

Exercises

8. The graph of $y = \tan\theta$ is shown. Use the method explained in the previous example to sketch the graph of $y = \cot\theta$.



9. Compare and contrast the graphs of $y = \tan\theta$ and $y = \cot\theta$.

EXERCISE 4d SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics: Principles and Process 12: p. 207 (Questions 1-4)
Holtmath 12: p. 298 (Questions 3, 8)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. It is important for you to know what the graphs of the basic trigonometric functions look like and what some of their properties are in order to graph transformations. This will be done in **Lesson 5: Investigating the Properties of *a*, *b*, *c* and *d*.**

Solutions

EXERCISE 4a

1. a. 0.7 b. −0.7 c. 0.7 d. −1

EXERCISE 4b

2.

Property	$y = \sin\theta$	$y = \cos\theta$
amplitude	1	1
maximum/minimum	1, −1	1, −1
period	2π	2π
y-intercepts	0	1
θ-intercepts	$n\pi, n \in I$	$\frac{\pi}{2} + n\pi, n \in I$

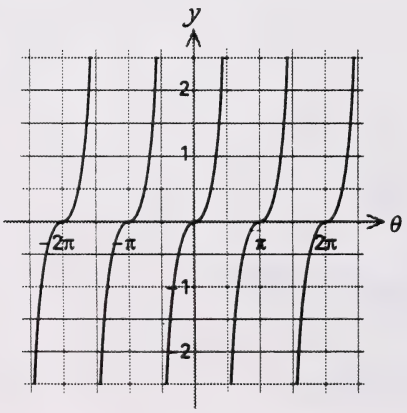
3. A sinusoidal curve is one that resembles that of the sine function.

4. Sound waves, electricity, ocean tides, etc.

EXERCISE 4c

5.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	undef	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	undef	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	0



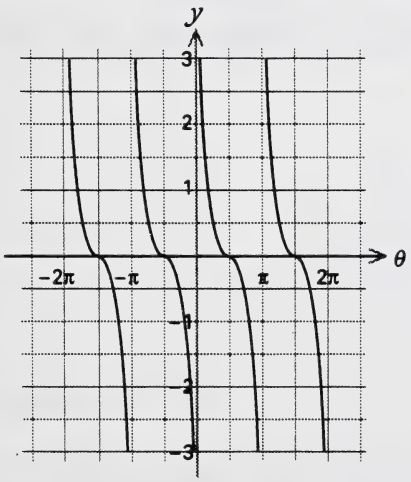
6.

Property	$y = \tan\theta$
amplitude	none
maximum/minimum	π
period	none
y-intercepts	0
θ-intercepts	$n\pi,\ n \in I$

7. a. T b. F c. T d. T e. T f. F g. T

EXERCISE 4d

8. $y = \cot\theta$



9. Both $y = \cot\theta$ and $y = \tan\theta$ have no amplitude and no maximum or minimum. They have the same period and the same general shape; however, the graph of $y = \cot\theta$ falls to the right, while the graph of $y = \tan\theta$ rises to the right. The θ-intercepts of $y = \tan\theta$ are $n\pi,\ n \in I$, and the θ-intercepts of $y = \cot\theta$ are $\frac{\pi}{2} + n\pi,\ n \in I$.

The y-intercept of $y = \tan\theta$ is at 0, while the y-intercept of $y = \cot\theta$ is undefined.

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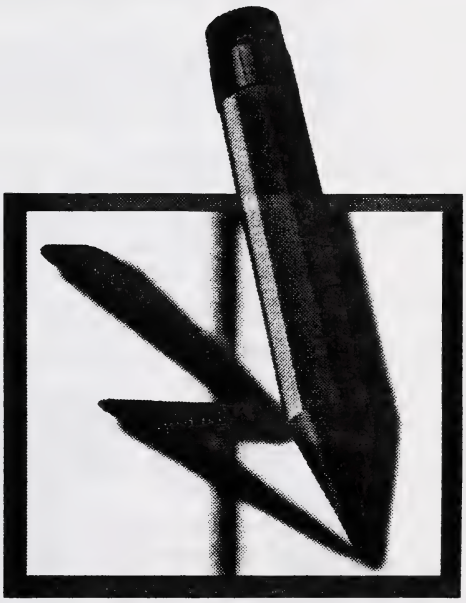
MATH FACTOR

TRIGONOMETRIC AND CIRCULAR FUNCTIONS



Lesson 5:
Investigating the Properties of *a*, *b*, *c* and *d*

Student Workbook

Produced by

THE EDUCATION STATION

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This guide accompanies **Investigating the Properties of *a*, *b*, *c* and *d*** from the Math Factor series.

Writer: Shauna Boyce
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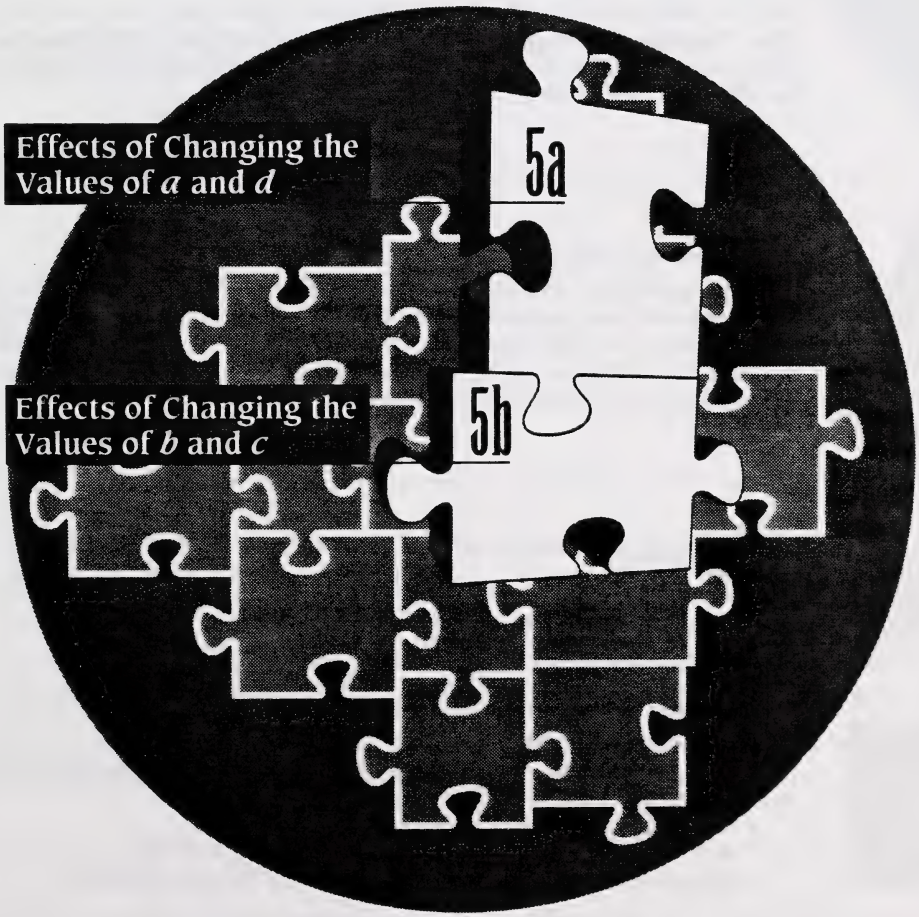
Lesson 5: INVESTIGATING THE PROPERTIES OF *a*, *b*, *c* and *d*

LEARNER EXPECTATIONS

By the end of Lesson 5 you should be able to:

- describe trigonometric transformations of the sine and cosine functions
- describe the effects of these transformations on the properties of the graphs
- describe transformations given the equation of the trigonometric function or the graph
- write the equation of a sine or cosine function where the graph has been given

This lesson has two parts:



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Building Blocks

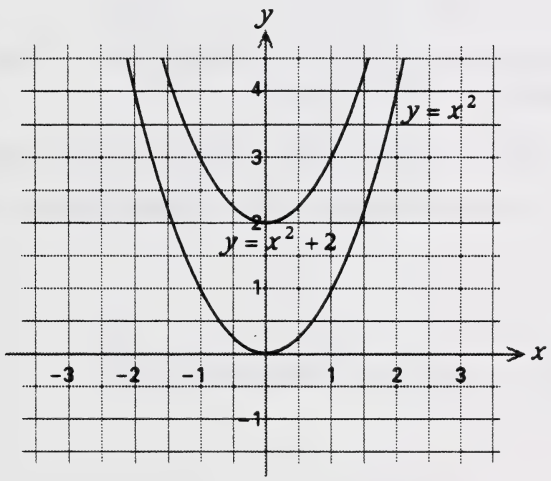
Do you remember how to transform functions? Before watching this program, refresh your memory by studying the following and doing the exercises.

Use the graph of $y = f(x)$, where $f(x) = x^2$, to show the transformations of functions you have studied in other math courses. The equation $y = f(x) + d$, where d is any real number, describes a vertical translation. The vertical translation of $y = f(x)$ will be up if d is a positive number and down if d is negative.

Example 1

Sketch the graph of $y = x^2 + 2$ on the same axis as the graph of $y = x^2$.

Solution



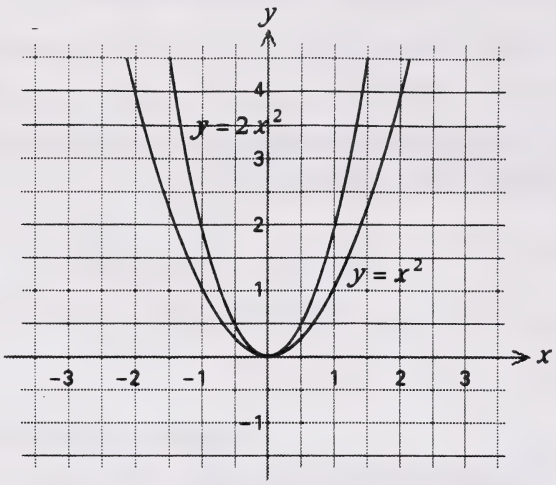
The result is a vertical translation two units up. For example, the point $(1, 1)$ on $y = x^2$ is translated to $(1, 3)$ on $y = x^2 + 2$.

The equation $y = af(x)$ describes a vertical expansion, a compression, or a reflection of the graph $y = f(x)$. The y-coordinate of each point on the graph is multiplied by a. If $a > 1$, it is an expansion; if $0 < a < 1$, it is a compression. If $a < 0$, the graph is also reflected about the x-axis.

Example 2

Sketch the graph of $y = 2x^2$ on the same graph as $y = x^2$.

Solution



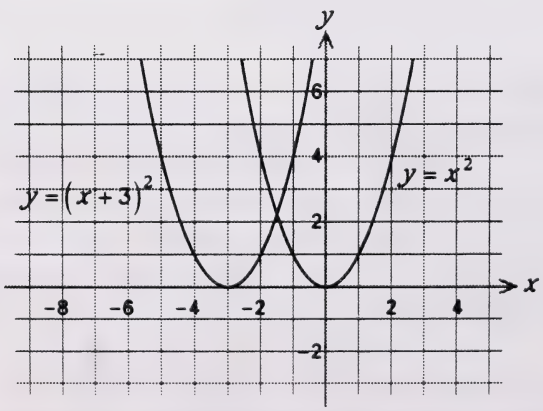
The result is a vertical expansion by a factor of 2. For example, the point $(1, 1)$ on $y = x^2$ is transformed to $(1, 2)$ on $y = 2x^2$.

The equation $y = f(x + c)$ describes a horizontal translation of c units of the graph of $y = f(x)$. The x-coordinate is increased by c if c is negative and decreased by c if c is positive.

Example 3

Sketch the graph of $y = (x + 3)^2$ on the same axis as $y = x^2$.

Solution



The result is a horizontal translation three units to the left. Each point on $y = x^2$ moved left three units. For example, $(-1, 1)$ is moved to $(-4, 1)$.

The equation $y = f(bx)$ describes a horizontal stretch or compression of the graph of $y = f(x)$. This transformation will be studied in Lesson 5b.

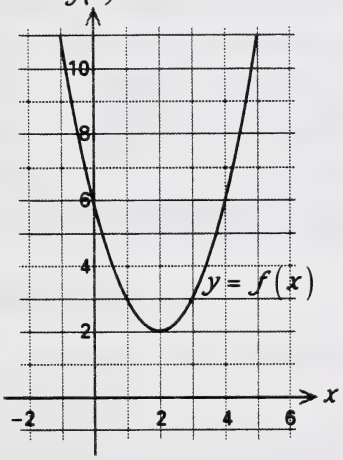
TEXT

Additional Examples
Mathematics 12: p. 4 (Example 1)
Mathematics: Principles and Process 12: p. 36 (Example 1), p. 39 (Example 1)
Holtmath 12: p. 31 (Example 1), p. 37 (Examples 1, 2), p. 39 (Examples 1, 2)

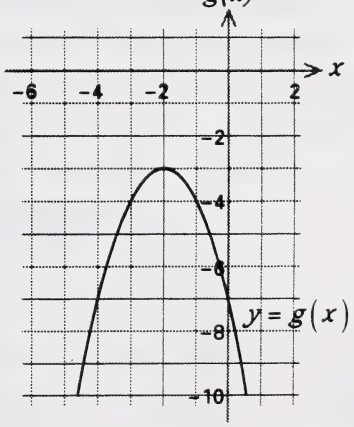
Exercises

1. The graphs of $y = f(x)$ and $y = g(x)$ shown below are transformations of $y = x^2$. Write the equation for each.

a.



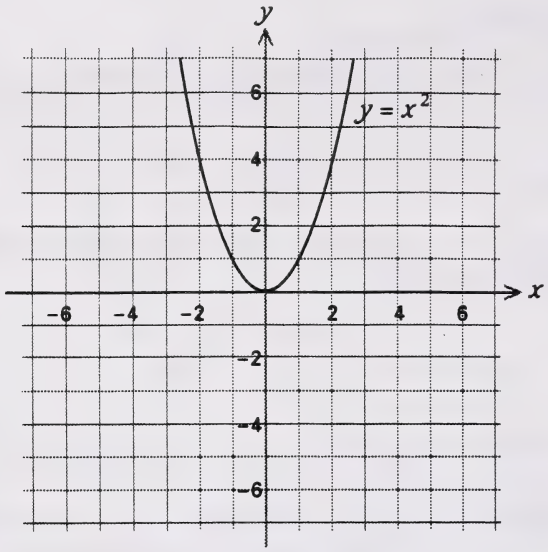
b.



2. State the domain and range of $f(x)$ and $g(x)$ from question 1.

3. a. The graph of $y = x^2$ is shown. Using the graph of $y = x^2$, sketch the graphs of each of the following polynomial functions.

- $y = -(x-4)^2 + 1$
- $y = 2(x-1)^2$



b. State the domain and range for each function in question 3a.

BUILDING BLOCKS SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics: Principles and Process 12: p. 36 (Questions 1-8), p. 40 (Questions 2-7)
Holtmath 12: p. 33 (Questions 2, 4), p. 38 (Questions 2, 4a, 4b, 5a, 5b), p. 41 (Questions 2, 4, 7)

Effects of Changing the Values of a and d

Challenge yourself

How does changing the values of a in $y = a\sin\theta$ and $y = a\cos\theta$ affect their graphs.

MATH FACTOR

Watch the Introduction and Video Segment 5a.

Changes in *a*

When $y = \cos\theta$ or $y = \sin\theta$ are multiplied by some number a, the result is a vertical expansion (if $a > 1$) or a compression (if $0 < a < 1$). Also, if a is negative, there is a reflection across the θ-axis. For example, if $y = 2\sin\theta$, the 2 stretches the graph of $y = \sin\theta$ by a factor of 2. Changes in a will affect only the range of the function and the amplitude of the graph. For $y = a\sin\theta$ or $y = a\cos\theta$, the amplitude of the graph is $|a|$. The range of the function is $-a \le f(\theta) \le a$. Each y-coordinate of $y = \sin\theta$ or $y = \cos\theta$ is multiplied by a in $y = a\sin\theta$ or $y = a\cos\theta$.

Changes in *d*

When $y = \sin\theta$ or $y = \cos\theta$ are increased or decreased by some number d, the result is a vertical translation up or down. For example, if $y = \cos\theta + 2$ is moved up two units, each y-coordinate of the graph of $y = \cos\theta$ will increase by two. Changes in d will affect the range of the function, the θ-intercepts and the y-intercept of the graph.

Example 1

Find the amplitude, vertical translation and equation of the horizontal axis of the graph of $y = f(\theta)$, where $f(\theta) = 3\sin\theta - 4$. Also determine the range of $f(\theta)$.

Solution

The amplitude is 3.

The vertical translation is 4 down.

The horizontal axis is halfway between the maximum and minimum value.

$$\frac{-7 + (-1)}{2} = -4$$

Therefore, $y = -4$ is the equation of the horizontal axis.

The maximum is $3 - 4 = -1$, and the minimum is $-3 - 4 = -7$. Therefore, $-7 \le f(\theta) \le -1$ is the range.

Example 2

Identify the amplitude, maximum and minimum values, θ-intercepts and y-intercepts of the graphs of $y = 2\cos\theta - 3$ and $y = \cos\theta$; then determine the domain and range of the functions $f(\theta) = 2\cos\theta - 3$ and $g(\theta) = \cos\theta$.

Solution

	$y = 2\cos\theta - 3$	$y = \cos\theta$
amplitude	2	1
maximum	$-1 \quad (2-3)$	1
minimum	$-5 \quad (-2-3)$	-1
θ-intercepts	none	$\ldots \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$
y-intercept	$-1 \quad (\text{let } \theta = 0)$	1
domain	$\{\theta \mid \theta \in R\}$	$\{\theta \mid \theta \in R\}$
range	$\{f(\theta) \mid -5 \le f(\theta) \le -1\}$	$\{g(\theta) \mid -1 \le g(\theta) \le 1\}$

Example 3

Find the equation of a sine function with a range of $\{f(\theta) \mid -4 \le f(\theta) \le 3\}$.

The formula to find amplitude is $\frac{\max - \min}{2}$.

Solution

From -4 to 3 is seven units. Therefore, the amplitude is $\frac{3-(-4)}{2} = 3.5$.

If $a = 3.5$, the maximum value would usually be at $y = 3.5$, but it is at $y = 3$. Therefore, the function has been moved down 0.5 units and, thus, d must equal -0.5.

Therefore, the equation of the sine function is $f(\theta) = 3.5\sin\theta - 0.5$.

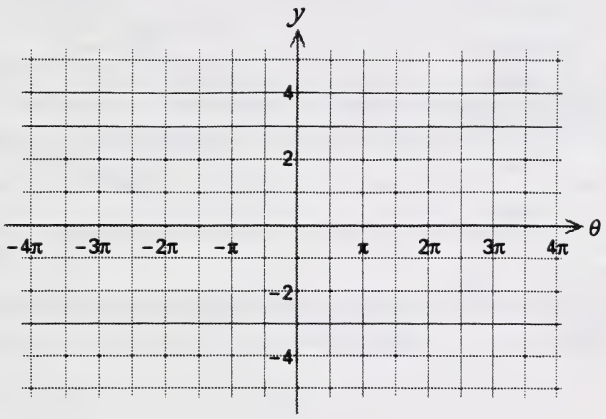
TEXT

Additional Examples
Mathematics 12: p. 229 (Examples 2, 3)
Mathematics: Principles and Process 12: p. 198 (Example 1)
Holtmath 12: p. 299 (Investigation 1), p. 302 (Investigation 1)

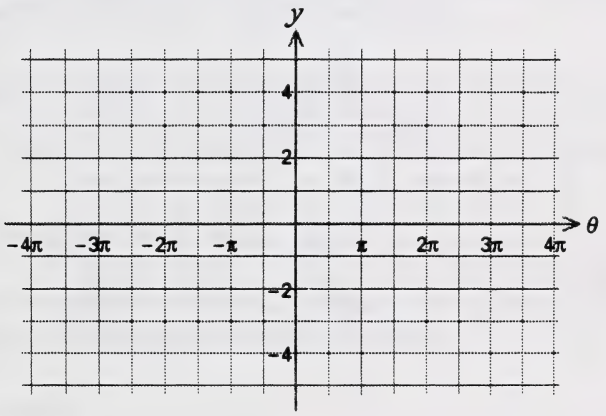
Exercises

1. Sketch the graphs of each of the following:

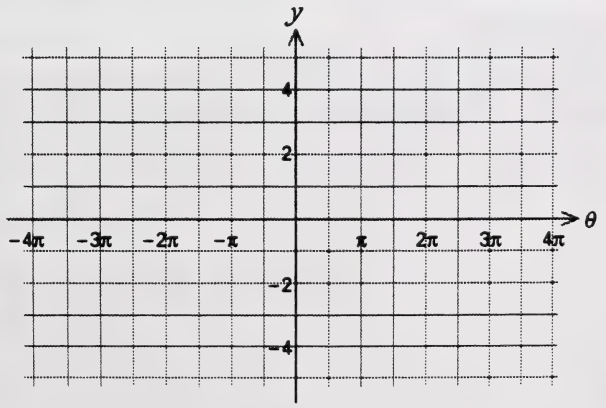
a. $y = \sin\theta - 1$



b. $y = 1.5\cos\theta$



c. $y = 3\cos\theta - 1$



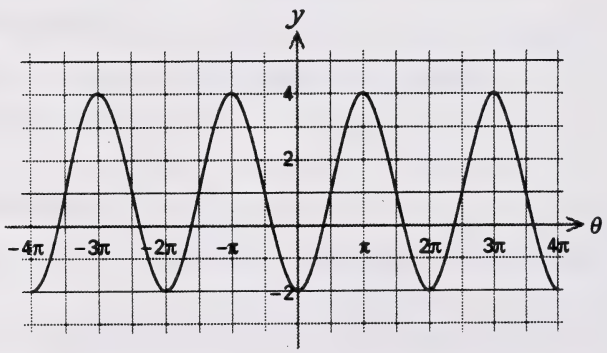
2. Identify the amplitude, maximum, minimum and y-intercepts of the graphs of $y = 3\cos\theta - 1$ and $y = \cos\theta$; then determine the domain and range of the functions $f(\theta) = 3\cos\theta - 1$ and $g(\theta) = \cos\theta$.

	$y = 3\cos\theta - 1$	$y = \cos\theta$
amplitude		
maximum		
minimum		
y-intercept		
domain		
range		

3. Find an equation of the graph of a sine function where the amplitude is 3 and the minimum value is -2.

EXCELLENCE

4. Find the equation of the graph of the cosine function shown.



EXERCISE 5a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: pp. 230-231 (Questions 1-6)
Mathematics: Principles and Process 12: pp. 195-197 (Explorations 1, 2)
Holtmath 12: p. 301 (Questions 4a, 4b, 4d, 7a, 7b, 7c, 8a)

Effects of Changing the Values of *b* and *c*

Challenge yourself

How does changing the values of b in $y = b\sin\theta$ and $y = b\cos\theta$ affect their graphs.

MATH FACTOR

Watch Video Segment 5b.

Example 1

The period of the graph of a sine or cosine function is $\frac{2\pi}{b}$.

Find an equation for the graph of a cosine function where the period is π and the maximum point is at $\left(-\frac{\pi}{6}, 1\right)$.

Solution

If the period is π, then $\frac{2\pi}{b} = \pi$, which means $b = 2$.

The graph of a cosine function usually has a maximum point at $(0, 1)$, so if it is now at $\left(-\frac{\pi}{6}, 1\right)$, then the graph has been shifted $\frac{\pi}{6}$ units to the left. This means that $c = +\frac{\pi}{6}$.

Therefore, the equation is $y = \cos 2\left(\theta + \frac{\pi}{6}\right)$.

$-\frac{\pi}{6}$ is called the phase shift. Here the phase shift is $\frac{\pi}{6}$ units to the left. The phase shift may change the θ-intercepts and the y-intercepts.

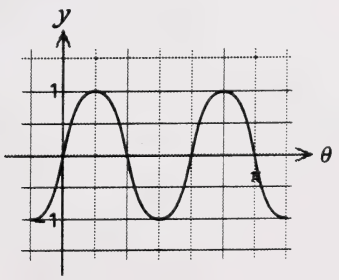
Example 2

Sketch the graph of $y = \sin(3\theta)$.

Solution

The parameter *b* affects the period of the graph. The 3 means that the graph will go through three cycles in a normal period of 2π instead of only one. Therefore, $\frac{2\pi}{3}$ is the period of this graph. One point on the horizontal axis will be at the origin, and the next θ-intercept will be at $\frac{2\pi}{3}$.

The period of the graph of the function is $\frac{2\pi}{3}$ or $\frac{360^\circ}{3} = 120^\circ$.



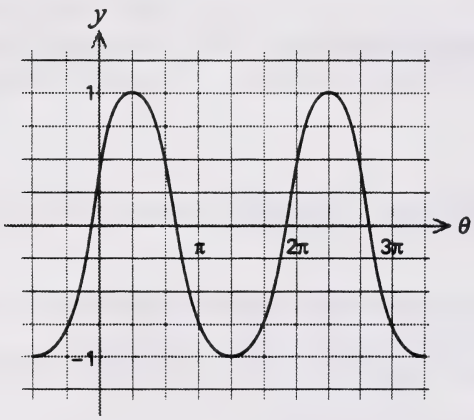
Example 3

Sketch the graph of $y = \cos\left(\theta - \frac{\pi}{3}\right)$.

Solution

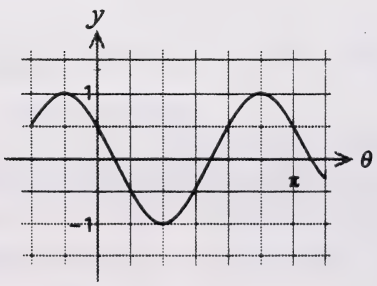
This graph is the graph of $y = \cos\theta$ shifted $\frac{\pi}{3}$ units to the right. In the graph of $y = \cos\theta$, there is a maximum value at $(0, 1)$ and again at $(2\pi, 1)$. In the graph of $y = \cos\left(\theta - \frac{\pi}{3}\right)$, the maximum points are now at $\left(\frac{\pi}{3}, 1\right)$ and $\left(\frac{7\pi}{3}, 1\right)$.

Every point on the graph of $y = \cos\theta$ moves $\frac{\pi}{3}$ rad to the right.



Example 4

The following graph shows a portion of the graph of a cosine function. Write an equation that could be used to describe this graph.



Solution

In comparison to the graph of $y = \cos\ \theta$, every point on the graph has been moved $\frac{\pi}{6}$ units to the left. $y = \cos\ \theta$ usually has a maximum point at $(0, 1)$, but now that point is at $\left(-\frac{\pi}{6}, 1\right)$. This means that $c = \frac{\pi}{6}$. Instead of the regular 2π rad required to complete one cycle, this graph only needs π rad. The next maximum point after $-\frac{\pi}{6}$ is at $\frac{5\pi}{6}$.

Therefore, the value required for b to make $\frac{2\pi}{b} = \pi$ is $b = 2$.

Therefore, the equation is $y = \cos\ 2\left(\theta + \frac{\pi}{6}\right)$.

TEXT

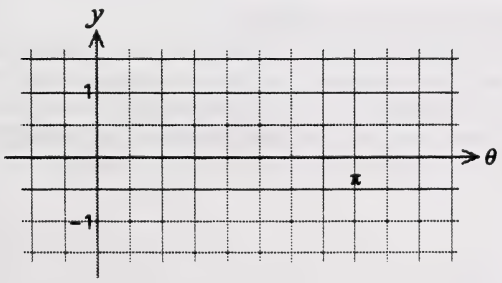
Additional Examples
Mathematics 12: p. 234 (Example 1), p. 235 (Example 2), p. 240 (Example 1), p. 241 (Example 2)
Mathematics: Principles and Process 12: p. 198 (Example 2)
Holtmath 12: p. 299 (Investigation 2), p. 302 (Investigation 2)

Exercises

5. List the similarities and differences between the graphs of $y = \sin\ \theta$ and $y = \sin\ 3\left(\theta - \frac{\pi}{3}\right)$.

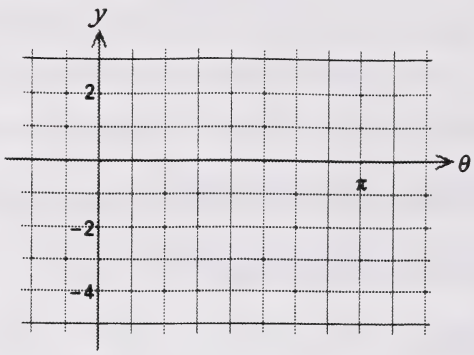
6. Sketch the graph of each of the following:

   a. $y = \sin\ 4(\theta + \pi)$



EXCELLENCE

b. $y = 3\sin 4(\theta + \pi) - 1$



7. For the function $y = f(\theta)$ where $f(\theta) = a\sin b(\theta + c) + d$, find the parameters a, b, c and d if the range is $-4 \le y \le 2$, the period of the graph is 4π and the phase shift is $\frac{\pi}{6}$ to the right.

EXERCISE 5b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: pp. 235-236 (Questions 1-8), pp. 241-242 (Questions 1-5)
Mathematics: Principles and Process 12: pp. 196-197 (Explorations 3, 4, 6.18, 6.19)
Holtmath 12: p. 304 (Questions 2, 3, 5a, 5b, 5d, 6, 7, 8a, 8b, 8d, 8e, 10)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. It is important that you understand the effects of changing the parameters *a*, *b*, *c* and *d* on the graphs of trigonometric functions. Future lessons will deal with the applications of trigonometric functions.

Solutions

BUILDING BLOCKS

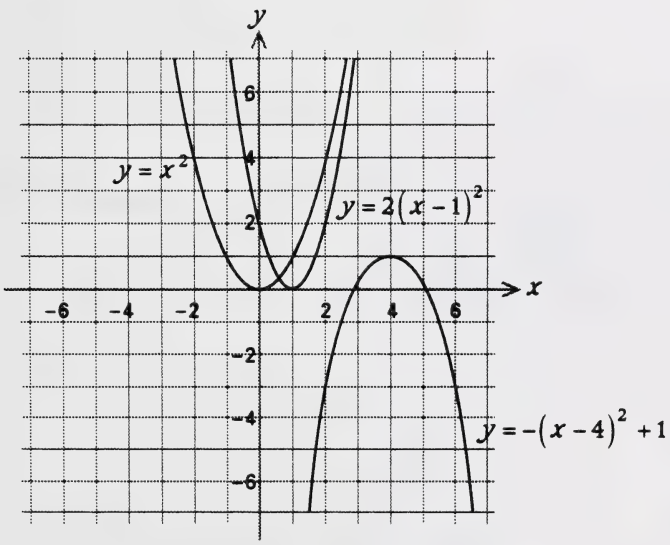
1. a. $f(x)=(x-2)^2+2$

   b. $g(x)=-1(x+2)^2-3$

2.

	$f(x)$	$g(x)$
Domain	$x \in R$	$x \in R$
Range	$f(x) \geq 2$	$g(x) \leq -1$

3. a.

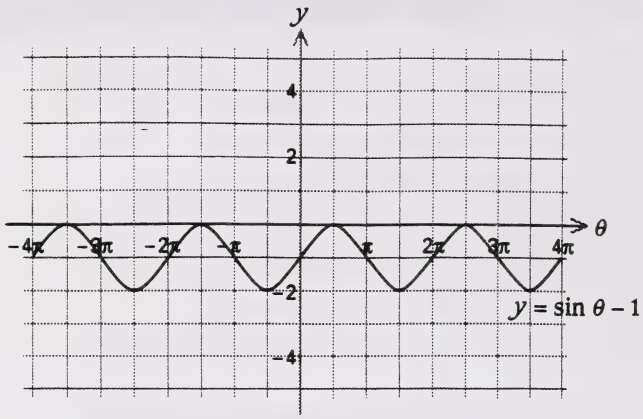


   b.

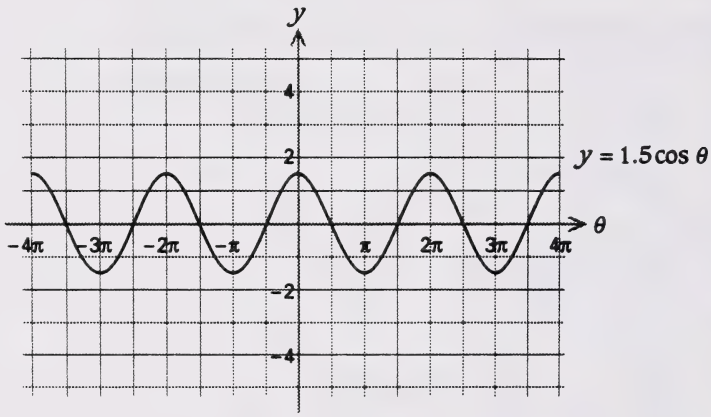
	$y=-(x-4)^2+1$	$y=2(x-1)^2$
Domain	$x \in R$	$x \in R$
Range	$y \leq 1$	$y \geq 0$

EXERCISE 5a

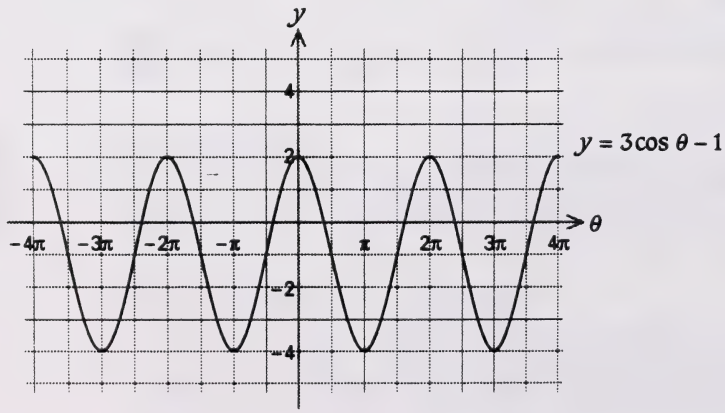
1. a.



b.



c.



2.

	$y = 3\cos\ \theta - 1$	$y = \cos\ \theta$
amplitude	3	1
maximum	2	1
minimum	-4	-1
y-intercept	2	1
domain	$\{\theta \mid \theta \in R\}$	$\{\theta \mid \theta \in R\}$
range	$\{-4 \le f(\theta) \le 2\}$	$\{-1 \le g(\theta) \le 1\}$

3. $y = 3\sin\ \theta + 1$

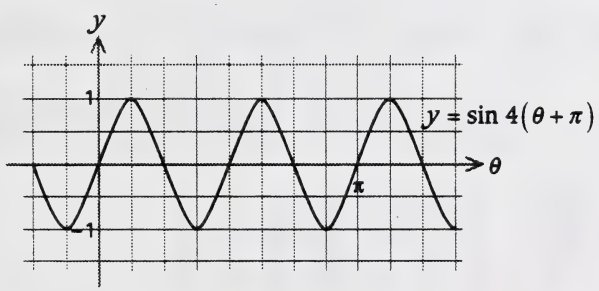
4. $y = -3\cos\ \theta + 1$

EXERCISE 5b

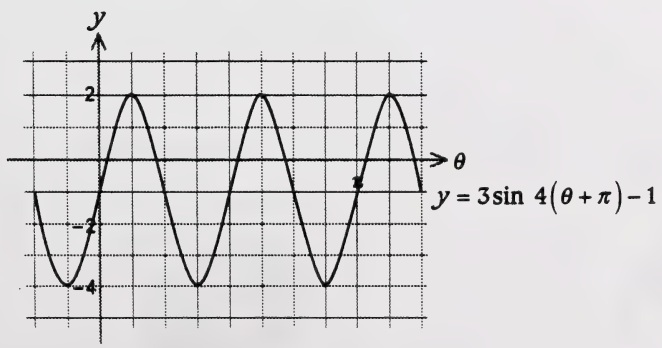
5. Similarities:
   - Both are sinusoidal functions.
   - Both have an amplitude of 1 and both horizontal axes are on the x-axis.

   Differences:
   - The period of $y = \sin\ \theta$ is 2π, while the period of $y = \sin\ 3\left(\theta - \frac{\pi}{3}\right)$ is $\frac{2\pi}{3}$.
   - The graph of $y = \sin\ 3\left(\theta - \frac{\pi}{3}\right)$ is the graph of $y = \sin\ \theta$ translated $\frac{\pi}{3}$ units to the right.

6. a.



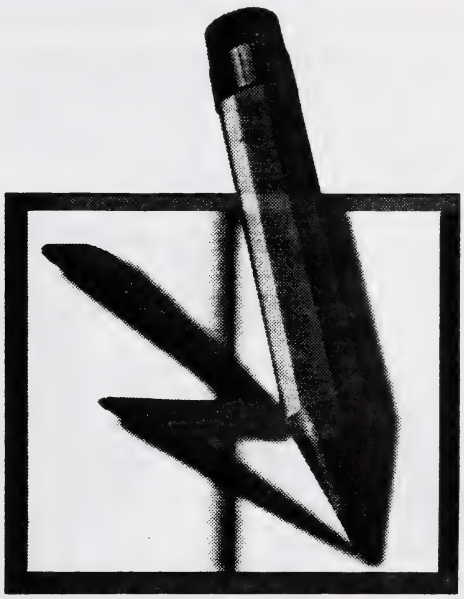
b.



7. $y = 3\sin\ \frac{1}{2}\left(\theta - \frac{\pi}{6}\right) - 1$

MATH FACTOR

TRIGONOMETRIC AND CIRCULAR FUNCTIONS



Lesson 6: Combined Transformations of Trigonometric Functions

Student Workbook

Produced by

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This guide accompanies **Combined Transformations of Trigonometric Functions** from the Math Factor series.

Writer: Shauna Boyce
Editor: Wilma Gonsalves
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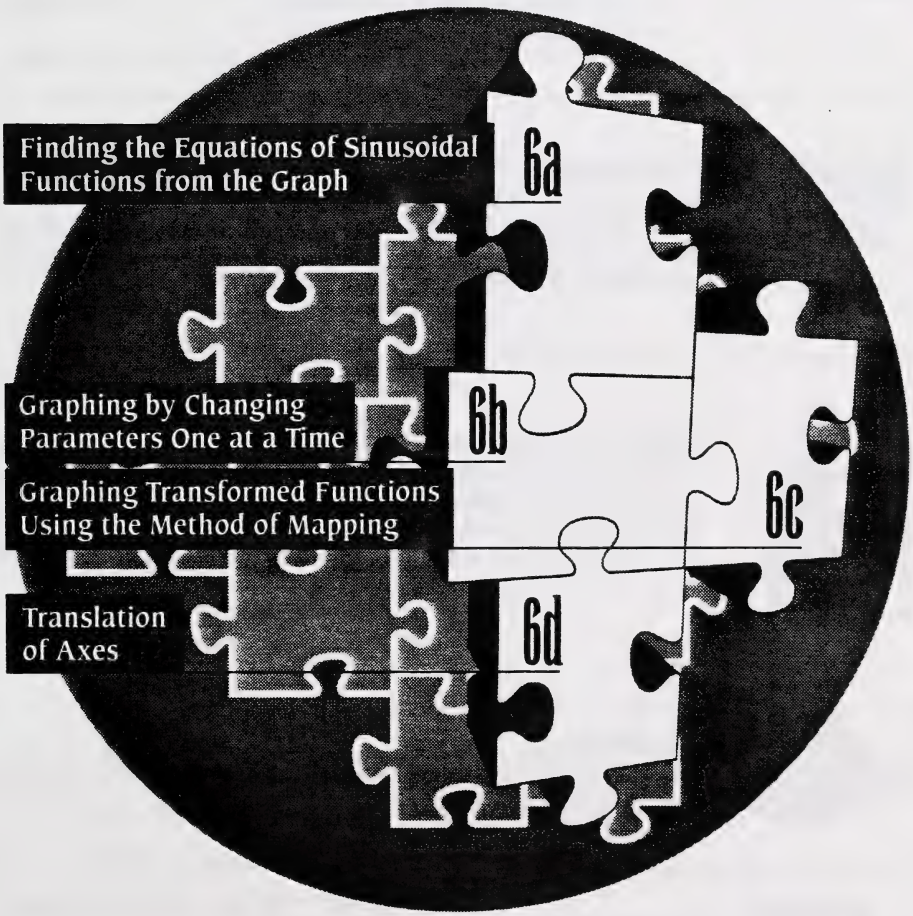
Lesson 6: COMBINED TRANSFORMATIONS OF TRIGONOMETRIC FUNCTIONS

LEARNER EXPECTATIONS

By the end of Lesson 6 you should be able to:

- recognize that the graph of a sinusoidal function can be represented by several different equations
- graph any transformed sinusoidal function using a variety of methods

This lesson has four parts:


Finding the Equations of Sinusoidal Functions from the Graph
6a
Graphing by Changing Parameters One at a Time
6b
Graphing Transformed Functions Using the Method of Mapping
6c
Translation of Axes
6d

EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The building blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to:

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. Sometimes the example is already written in the workbook for your convenience. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

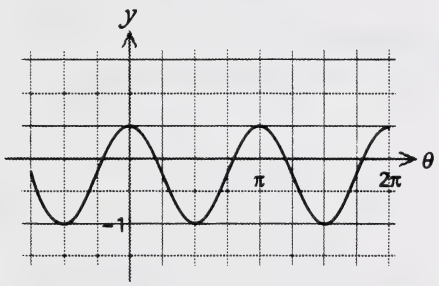
- *Mathematics 12–Addison Wesley*
- *Mathematics: Principles and Process 12*
- *Holt Math 12*

Finding the Equations of Sinusoidal Functions from the Graph

Watch the Introduction and Video Segment 6a.

Example

Determine the equation of the graph of this sine function.



Solution

Determine values for the parameters a, b, c and d.

Step 1: To find values for a and d, determine the range.

- The maximum is at $\frac{1}{2}$.
- The minimum is at -1.

From -1 to $\frac{1}{2}$ is $1\frac{1}{2}$ units, and half of this is $\frac{3}{4}$. Therefore, $a = \frac{3}{4}$. Also, $\frac{3}{4}$ unit up from -1 or down from $\frac{1}{2}$ is $-\frac{1}{4}$, so the equation of the horizontal axis is at $y = -\frac{1}{4}$. Therefore, $d = -\frac{1}{4}$. It may be helpful for you to draw this horizontal axis.

Step 2: Determine the value of the parameter c.
Sketch the graph of $y = \sin\theta$ on the axis. Find a point on $y = \sin\theta$ and observe how this point has shifted to a point on the transformed graph. For example, the point at $\theta = 0$ on the graph of $y = \sin\theta$ is at $\theta = -\frac{\pi}{4}$ on the transformed graph. The graph of $y = \sin\theta$ has moved $\frac{\pi}{4}$ rad to the left; therefore, $c = \frac{\pi}{4}$.

Step 3: Determine the value of the parameter b.
If you move to the right from $\theta = 0$, the next maximum point occurs when $\theta = \pi$, which is π units away. This means that the period of this function is π units. Since $\frac{2\pi}{b}$ = period, $\frac{2\pi}{b} = \frac{\pi}{1}$, therefore $b = 2$.

Put these steps together to determine the equation of the function.

$$y = \tfrac{3}{4}\sin 2\left(\theta + \frac{\pi}{4}\right) - \tfrac{1}{4}$$

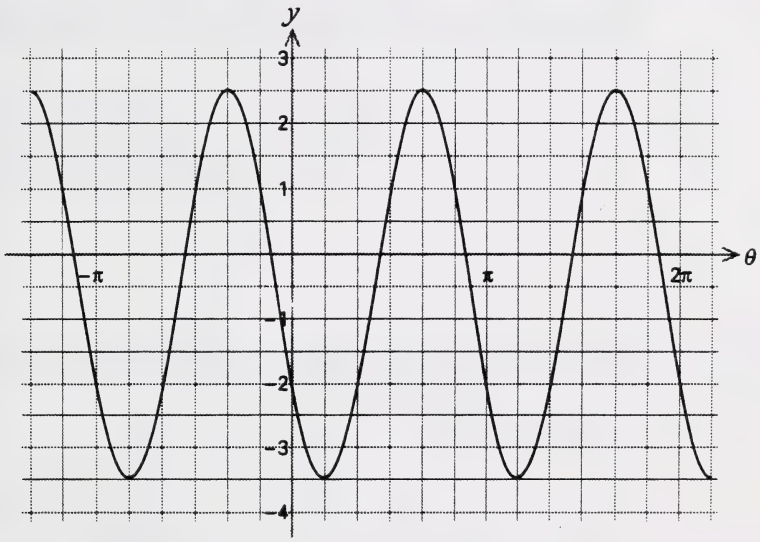
There are other possible equations that could represent this graph. For example, you could add or subtract multiples of the period to the phase shift, $-\frac{\pi}{4}$. Two other possible equations are $y = \frac{3}{4}\sin 2\left(\theta - \frac{3\pi}{4}\right) - \frac{1}{4}$ and $y = \frac{3}{4}\sin 2\left(\theta - \frac{5\pi}{4}\right) - \frac{1}{4}$. The equation could also be represented by a cosine function.

Additional Examples
Mathematics 12: p. 246 (Example 2)
Mathematics: Principles and Process 12: p. 208 (Example)

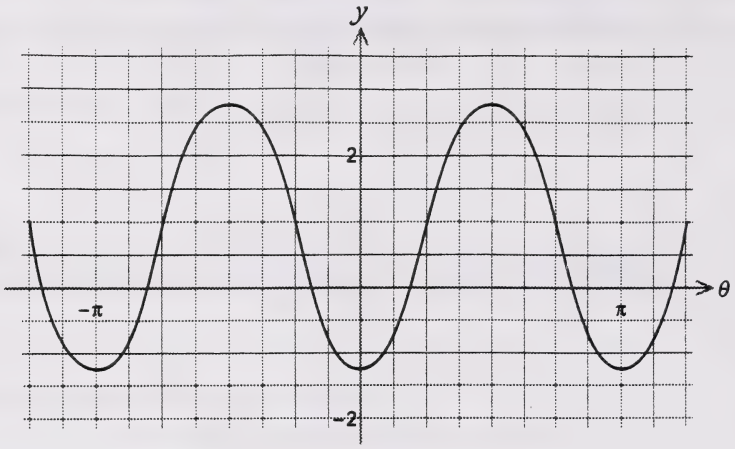
Exercises

1. Determine the equation of a cosine function that could describe the graph shown in the example.

2. Determine the equation of a cosine function that could describe the following graph.

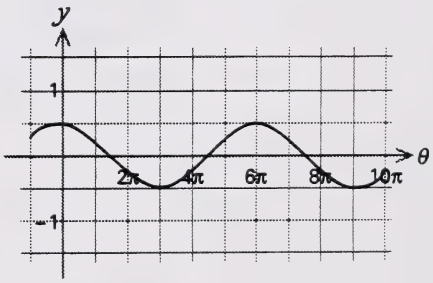


3. Determine the equation of a sine function that could describe the following graph.



4. Express the equation that describes the graph below as follows:

a. a cosine function



b. a sine function

5. An equation where the value of a in the equation $y = a \sin b(\theta + c) + d$ is negative can also be used to describe the graph in question 4. Determine such an equation.

EXERCISE 6a SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 247 (Questions 1, 2)
Mathematics: Principles and Process 12: pp. 208-209 (Questions 1-5)
Holtmath 12: p. 304 (Question 10)

Graphing by Changing Parameters One at a Time

Challenge yourself

Use transformations to help you sketch the graph of $y = \frac{3}{4}\cos 2\left(\theta - \frac{\pi}{4}\right) - \frac{1}{4}$.

MATH FACTOR

Watch Video Segment 6b.

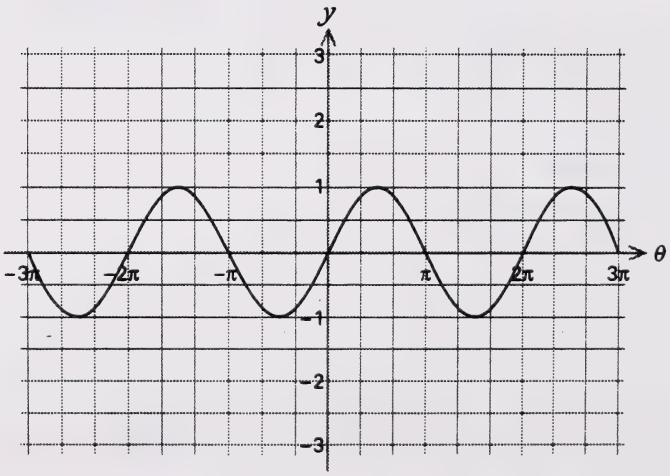
Example

Sketch the graph of the function $f(\theta) = 2\sin\frac{1}{2}\left(\theta - \frac{\pi}{3}\right) - 2$.

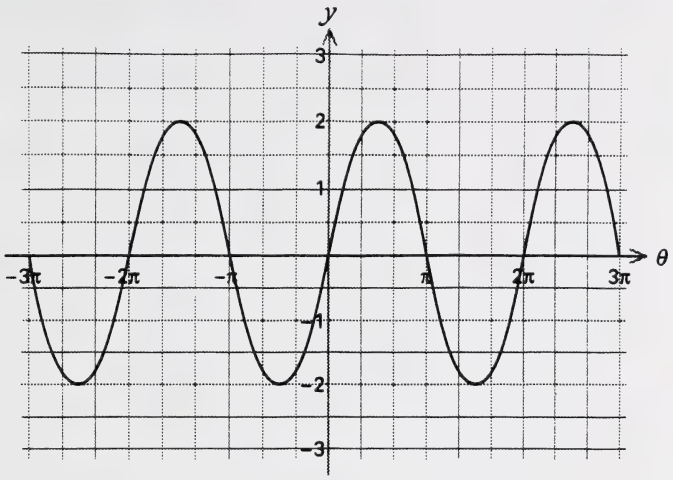
Solution

To sketch the graph of $y = f(\theta)$, work through the following steps.

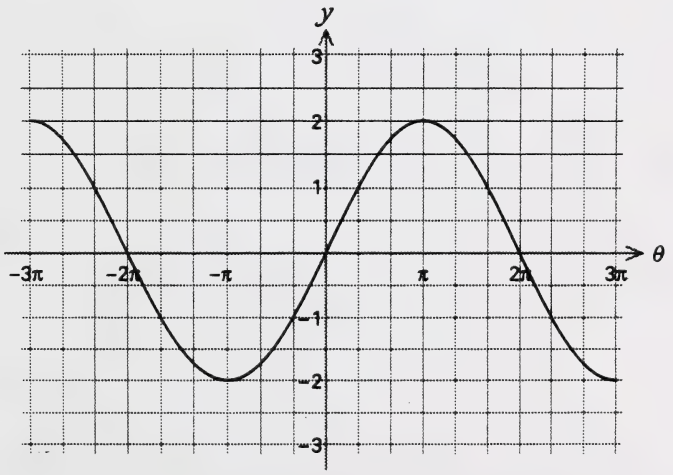
Step 1: Graph $y = \sin\theta$.



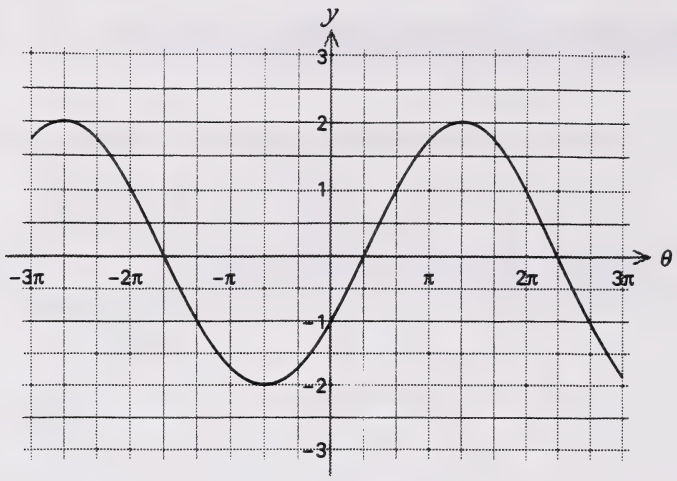
Step 2: Graph $y = 2\sin\theta$.



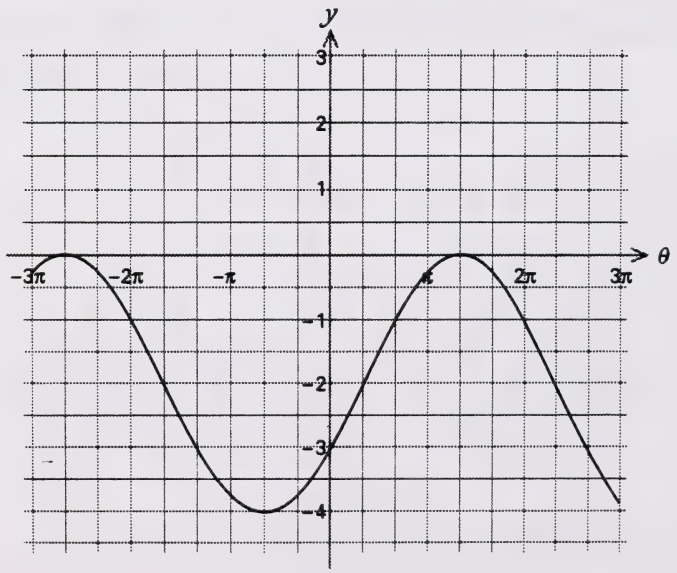
Step 3: Change the period and draw $y = 2\sin\frac{1}{2}(\theta)$.



Step 4: Move the graph $\frac{\pi}{3}$ rad to the right for the graph of $y = 2\sin\frac{1}{2}\left(\theta - \frac{\pi}{3}\right)$.



Step 5: Shift the graph down two units to sketch the graph of $y = 2\sin\frac{1}{2}\left(\theta - \frac{\pi}{3}\right) - 2$.



In general, to graph $y = a\sin b(\theta + c) + d$, draw the graph of $y = a\sin b(\theta)$; then, translate it c units to the right or left and d units up or down.

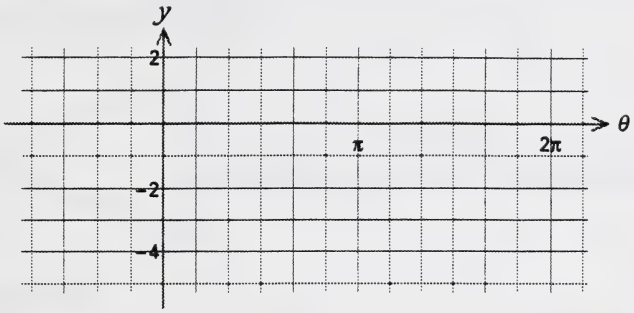
TEXT

Additional Examples
Mathematics 12: p. 240 (Examples 1, 2)
Mathematics: Principles and Process 12: p. 202 (Example)
Holtmath 12: p. 300 (Example), p. 303 (Example)

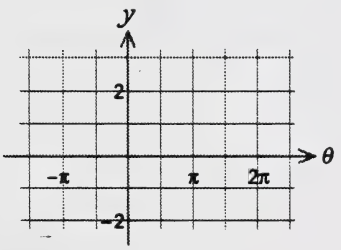
Exercises

Using the method outlined in the example, sketch the graph of each of the following trigonometric functions.

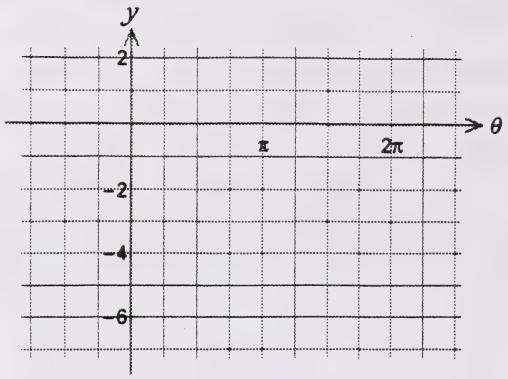
6. $y = F(\theta)$, where $F(\theta) = 2\sin 4\left(\theta + \frac{\pi}{6}\right) - 2$



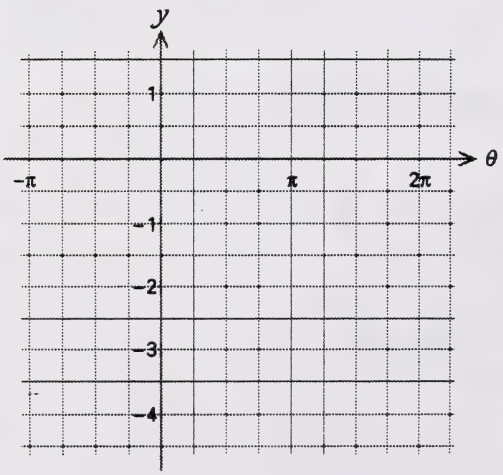
7. $y = G(\theta)$, where $G(\theta) = \cos 2(\theta - \pi) + 1$



8. $y = H(\theta)$, where $H(\theta) = 3\sin\left(\theta + \frac{\pi}{4}\right) - 3$



9. $y = K(\theta)$, where $K(\theta) = -2\cos(\theta + \pi) - 2$



EXERCISE 6b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 248 (Question 6), p. 249 (Question 9)
Mathematics: Principles and Process 12: p. 203 (Questions 3, 4, 6-13)
Holtmath 12: p. 301 (Questions 7-10, 12), p. 304 (Questions 2-5, 8)

Graphing Transformed Functions Using the Method of Mapping

Challenge yourself

Given $f(\theta) = \sin\theta$ and $g(\theta) = 3\sin 2\left(\theta - \frac{\pi}{4}\right) + 1$, how can mapping be used to help sketch the graph of $g(\theta)$?

Watch Video Segment 6c.

Example

Sketch the graph of $y = 2\sin 2\left(\theta - \frac{\pi}{4}\right) + 3$ by mapping values from $y = \sin\theta$.

Solution

Let y_0 = original y values.
Let θ_0 = original θ values.

$y_0 = \sin\theta_0 \qquad y = 2\sin 2\left(\theta - \frac{\pi}{4}\right) + 3 \quad \text{or} \quad \frac{y-3}{2} = \sin 2\left(\theta - \frac{\pi}{4}\right)$

$$y_0 \to \frac{y-3}{2}$$
$$2y_0 \to y - 3$$
$$y \to 2y_0 + 3$$

Solve for y_0.

The value of y in the transformed function is twice the orginal y value plus three.

$$\theta_0 \to 2\theta - \frac{\pi}{2}$$
$$\left(\frac{\theta_0}{2} + \frac{\pi}{4}\right) \to \theta$$
$$\theta \to \left(\frac{\theta_0}{2} + \frac{\pi}{4}\right)$$

Solve for θ.

The value of θ in the transformed function is one half the original θ value plus $\frac{\pi}{4}$.

For example, the point $\left(\frac{\pi}{2}, 1\right)$ on $y = \sin\theta$ will be mapped on to $\theta = \frac{\frac{\pi}{2}}{2} + \frac{\pi}{4} = \frac{\pi}{2}$.

$y = 2(1) + 3$

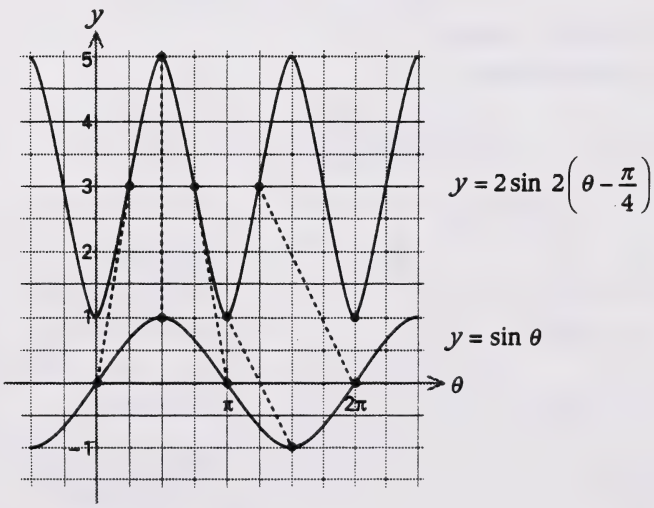
$y = 5$

Therefore, $\left(\frac{\pi}{2}, 1\right)$ is mapped to $\left(\frac{\pi}{2}, 5\right)$.

Table of Values:

θ_0	y_0
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	−1
2π	0

θ	y
$\frac{\pi}{4}$	3
$\frac{\pi}{2}$	5
$\frac{3\pi}{4}$	3
π	1
$\frac{5\pi}{4}$	3



Exercises

10. Start with $y = \cos\ \theta$ and graph $y = \frac{3}{4}\cos\ 2\left(\theta - \frac{\pi}{4}\right) - 1$ by applying the concept of mapping values. Complete the table of values before graphing. Graph the function on a piece of graph paper.

$y \rightarrow$ _____ $\theta \rightarrow$ _____

θ_0	y_0

θ	y

11. Using the mapping method, graph the function $f(\theta) = 3\cos\ \frac{1}{2}\left(\theta - \frac{\pi}{3}\right) + 1$.

EXERCISE 6c SOLUTIONS are at the back of this workbook.

Translation of Axes

Watch Video Segment 6d.

Example

Sketch the graph of $y = g(\theta)$, where $g(\theta) = 3\sin 2\left(\theta - \frac{\pi}{2}\right) - 1$.

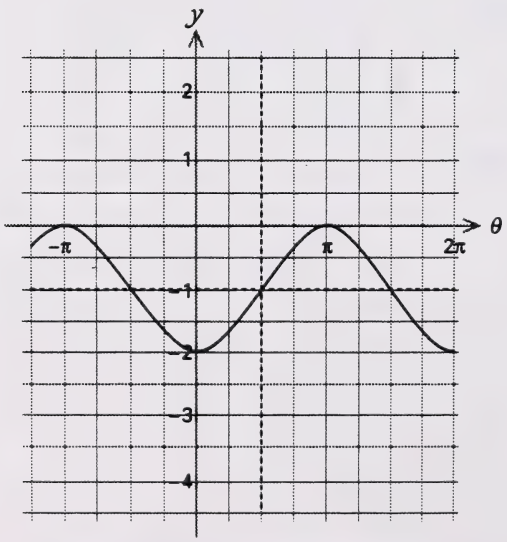
Solution

The parameters c and d will determine the horizontal and vertical movement of the graph of $y = \sin\theta$.

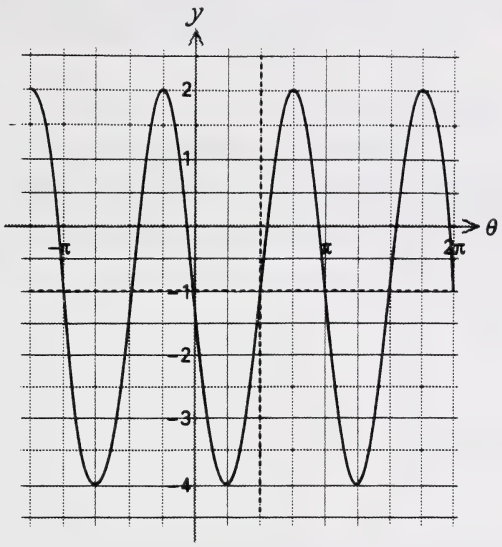
$c = -\frac{\pi}{2}$ and $d = -1$

Everything should translate $\frac{\pi}{2}$ rad to the right and one unit down.

The new origin would be at $\left(\frac{\pi}{2}, -1\right)$. The graph of $y = \sin\theta$ with axes of $y = -1$ and $x = \frac{\pi}{2}$ is shown.



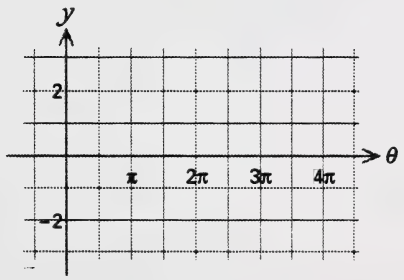
Now complete the graph by changing the period to π and the amplitude to 3.



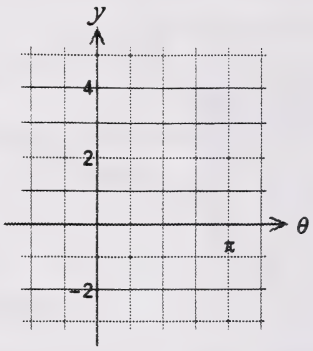
Exercises

Using the method of translating the axes, sketch the graph of each of the following:

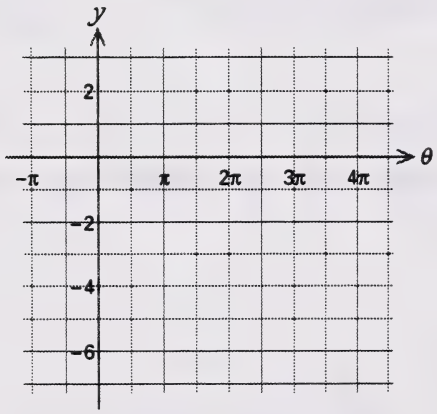
12. $y = 2\sin(\theta - \pi)$



13. $y = 3\cos 4\left(\theta - \frac{\pi}{2}\right) + 1$

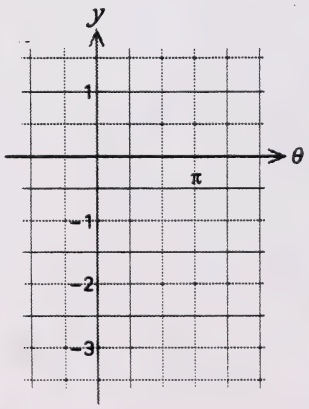


14. $y = 4\cos \frac{1}{2}\left(\theta + \frac{\pi}{4}\right) - 2$



EXCELLENCE

15. $y = -2\sin 3\left(\theta + \frac{\pi}{3}\right) - 1$



EXERCISE 6d SOLUTIONS are at the back of this workbook.

Additional Examples
Mathematics 12: p. 248-249 (Questions 6, 9)
Mathematics: Principles and Process 12: p. 203 (Question 7)
Holtmath 12: p. 304 (Questions 2, 3, 5, 8)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. Choose whichever method or combination of methods (of graphing) suits your purpose or is most comfortable for you to use.

Solutions

EXERCISE 6a

1. $y = g(\theta)$, where $g(\theta) = \frac{3}{4}\cos 2\theta - \frac{1}{4}$

2. $y = f(\theta)$, where $f(\theta) = 3\cos 2\left(\theta + \frac{\pi}{3}\right) - \frac{1}{2}$

   or

   $y = g(\theta)$, where $g(\theta) = 3\cos 2\left(\theta + \frac{2\pi}{3}\right) - \frac{1}{2}$

3. $y = f(\theta)$, where $f(\theta) = 2\sin 2\left(\theta - \frac{\pi}{4}\right) + \frac{3}{2}$

   or

   $y = g(\theta)$, where $g(\theta) = 2\sin 2\left(\theta - \frac{3\pi}{4}\right) + \frac{3}{2}$

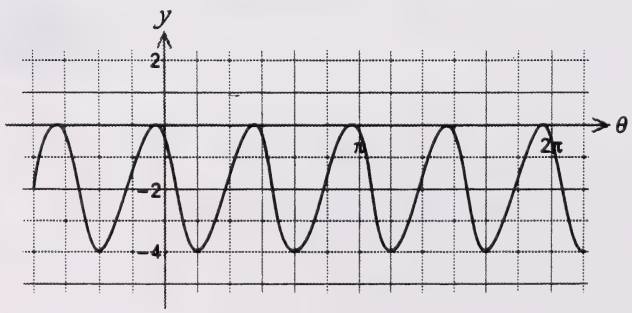
4. a. $y = f(\theta)$, where $f(\theta) = \frac{1}{2}\cos \frac{1}{3}\theta$

   b. $y = g(\theta)$, where $g(\theta) = \frac{1}{2}\sin \frac{1}{3}\left(\theta + \frac{3\pi}{2}\right)$

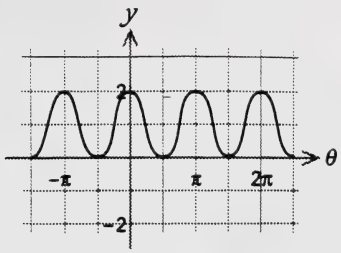
5. $y = f(\theta)$, where $f(\theta) = \frac{1}{2}\sin \frac{1}{3}\left(\theta - \frac{3\pi}{2}\right)$

EXERCISE 6b

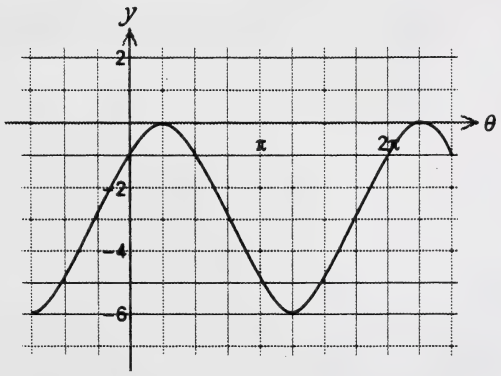
6. $y = 2\sin 4\left(\theta + \frac{\pi}{6}\right) - 2$



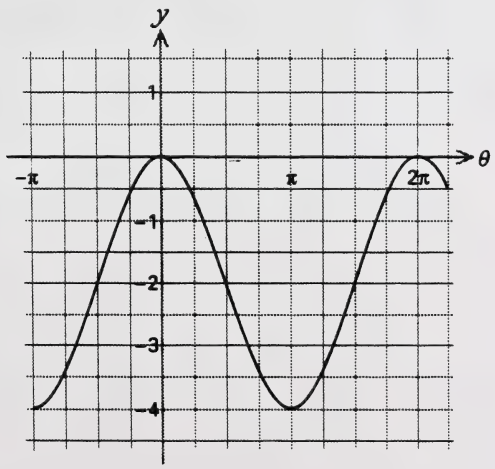
7. $y = \cos 2(\theta - \pi) + 1$



8. $y = 3\sin\left(\theta + \frac{\pi}{4}\right) - 3$



9. $y = -2\cos(\theta + \pi) - 2$

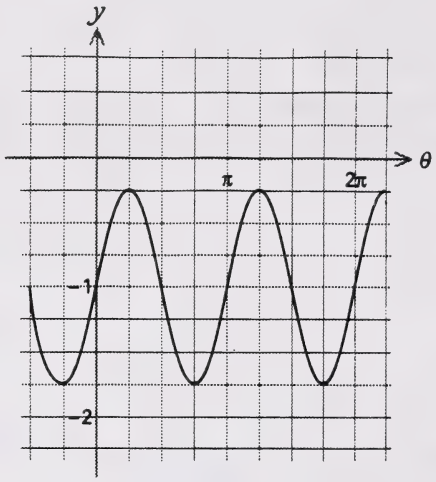


EXERCISE 6c

10. $y \to \frac{3y_0}{4} - 1$ $\quad \theta \to \frac{\theta_0}{2} + \frac{\pi}{4}$

θ_0	y_0
$-\frac{\pi}{2}$	0
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1

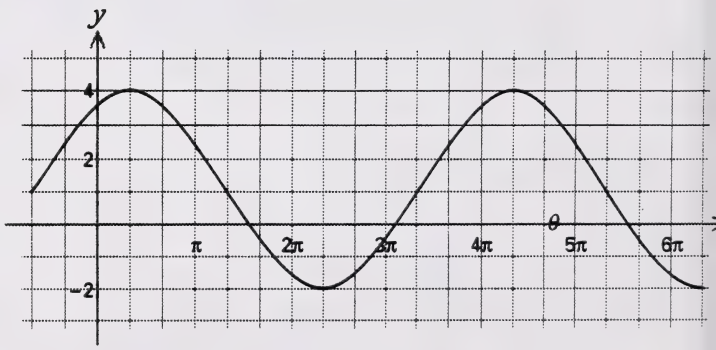
θ	y
0	-1
$\frac{\pi}{4}$	-0.25
$\frac{\pi}{2}$	-1
$\frac{3\pi}{4}$	-1.75
π	-1
$\frac{5\pi}{4}$	-0.25



11. $y \to 3y_0 + 1$ $\quad \theta \to 2\theta_0 + \frac{\pi}{3}$

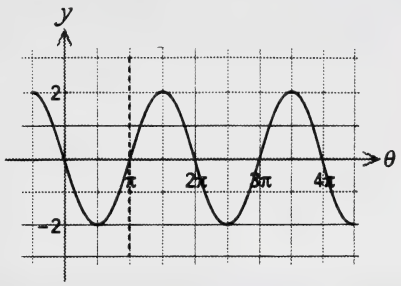
θ_0	y_0
$-\frac{\pi}{2}$	0
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1
$\frac{5\pi}{2}$	0

θ	y
$-\frac{2\pi}{3}$	1
$\frac{\pi}{3}$	4
$\frac{4\pi}{3}$	1
$\frac{7\pi}{3}$	-2
$\frac{10\pi}{3}$	1
$\frac{13\pi}{3}$	4
$\frac{16\pi}{3}$	1

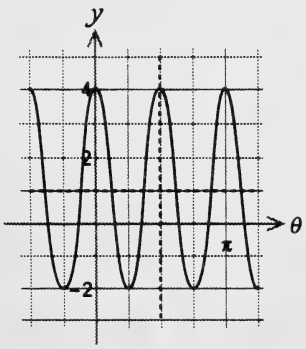


EXERCISE 6d

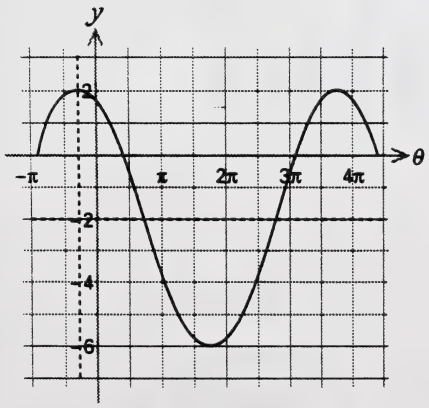
12. $y = 2\sin\left(\theta - \pi\right)$



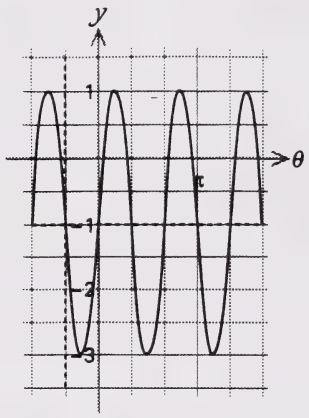
13. $y = 3\cos 4\left(\theta - \frac{\pi}{2}\right) + 1$



14. $y = 4\cos 0.5\left(\theta + \frac{\pi}{4}\right) - 2$



15. $y = -2\sin 3\left(\theta + \frac{\pi}{3}\right) - 1$



MATH FACTOR

TRIGONOMETRIC AND CIRCULAR FUNCTIONS

Lesson 7:
Building Trigonometric Models:
Graphing Trigonometric Functions

Student Workbook

Produced by

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This guide accompanies **Building Trigonometric Models: Graphing Trigonometric Functions** from the Math Factor series.

Writer: Shauna Boyce
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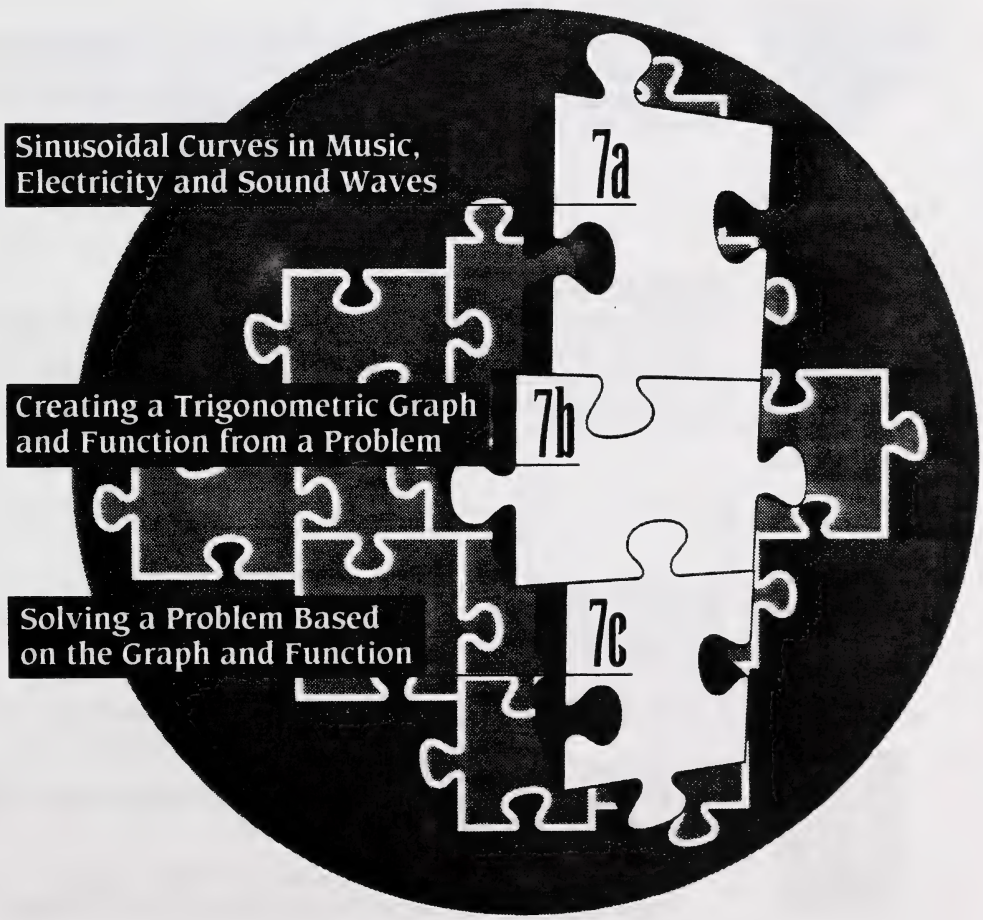
Lesson 7: BUILDING TRIGONOMETRIC MODELS: GRAPHING TRIGONOMETRIC FUNCTIONS

LEARNER EXPECTATIONS

By the end of Lesson 7 you should be able to:

- recognize that naturally occurring phenomena can be modelled with trigonometric functions
- recognize where and how mathematical models are built and used
- build and use a trigonometric function to solve problems involving real-world phenomena

This lesson has three parts:


Sinusoidal Curves in Music, Electricity and Sound Waves
7a
Creating a Trigonometric Graph and Function from a Problem
7b
Solving a Problem Based on the Graph and Function
7c

EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

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Textbook Reference:

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- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Sinusoidal Curves in Music, Electricity and Sound Waves

MATH FACTOR

Watch the Introduction and Video Segment 7a.

Examples

The examples for this section are included in the video. It is important to recognize that each of the three parameters mentioned has a specific application in real-world phenomena.

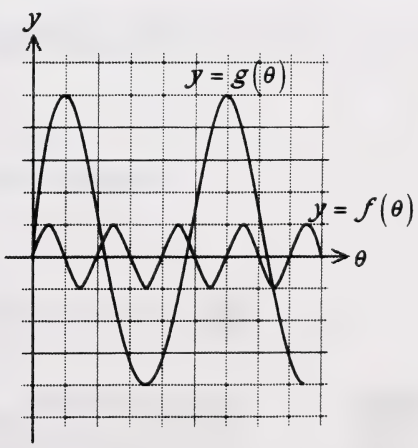
Exercises

1. List three other applications of sinusoidal waves that occur in the real world.

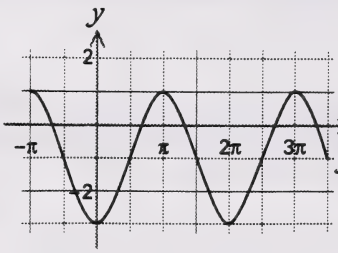
2. The graph to the right is of two different sound waves. The vertical axis represents the loudness, and the horizontal axis represents the wavelength.

   a. Which sound wave is louder? Explain.

   b. Which has the higher pitch? Explain.



3. The definition given in the video for amplitude in segment 4b of Lesson 4 was "the vertical distance between the horizontal axis and the maximum or minimum point." In this segment, the definition "the maximum value minus the minimum value, divided by 2" was also given. Using the graph, explain how both definitions give the same result.



$y = a\sin(\theta + c) + d$

4. Using the information in question 3, answer the following:

   a. What is the period of the graph?

   b. Is this the graph of a sinusoidal function?

   c. What is the phase shift of $y = a\sin(\theta + c) + d$?

   d. What is the vertical translation?

   e. Using the formula, period $= \frac{2\pi}{b}$, calculate the value of b.

   f. Using the previous answers, find the equation of the sine function.

EXERCISE 7a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: pp. 247-248 (Questions 1, 3, 4)
Mathematics: Principles and Process 12: p. 195 (Questions 2, 3, 5, 8, 12)
Holtmath 12: p. 304 (Questions 2, 3, 5, 7, 8)

Creating a Trigonometric Graph and Function from a Problem

Challenge yourself

Graph a trigonometric function that represents the path that a passenger on a Ferris wheel will travel.

MATH FACTOR

Watch Video Segment 7b.

Example 1

A Ferris wheel has a radius of 15 m and rotates once every 30 s. Passengers get on at a platform 1 m above the ground. Draw a graph to show the heights that a passenger will reach during the first two rotations; then write the equation of a cosine function that could describe this graph.

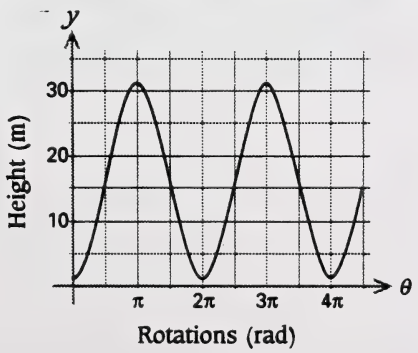
Let the vertical axis represent the height of the chair above the ground and the horizontal axis represent the angle of rotation of the wheel.

Solution

Since a passenger will get on 1 m above the ground, one point on the graph will be at $(0, 1)$.

The maximum height the passenger will reach is 30 m + 1 m = 31 m. The median height will be halfway in between (or the radius plus 1 m); therefore, the median height is 16 m.

Since one full rotation is 2π rad and the passenger will be back at a height of 1 m once the Ferris wheel has completed one full rotation, another point on the graph will be at $(2\pi, 1)$.



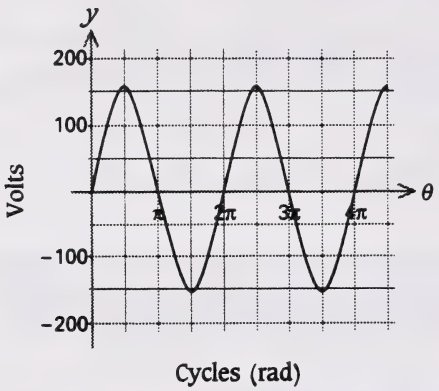
The amplitude is 15, the period is 2π and the graph is translated 16 units up. If this is a cosine function, it has been shifted π units to the right of its original position. Therefore, the function could be given as $y = f(\theta)$ where $f(\theta) = 15\cos(\theta - \pi) + 16$.

Example 2

Normal 110 V household alternating current varies from -155 V to 155 V with a frequency of 60 Hz. One hertz (Hz) is one cycle per second. Draw a graph using 2π as the period of the function, showing at least two cycles; then write the equation of a sine function that could represent this graph.

Note: To simplify the equation, the horizontal axis represents cycles (in rads). It could have also represented time, in which case the equation would have to be adjusted.

Solution



The parameters of the graph are $a = 155$, $b = 1$, $c = 0$ and $d = 0$. Therefore, the sine function can be expressed by $y = f(\theta)$, where $f(\theta) = 155 \sin \theta$.

TEXT

Additional Examples
Mathematics 12: p. 258-260 (Example 1), p. 259 (Example 2)
Holtmath 12: p. 305

Exercises

5. Sketch a graph and write the equation of a function to represent each of the following applications of trigonometric functions. Use 2π rad as the period of the graph.

   a, A windmill with a radius of 2.0 m makes one complete revolution every 8 s. The bottom of the wheel is 10 m above the ground. Let the vertical axis represent the height above the ground and the horizontal axis the angle of rotation (in rads).

   b. The rise and fall of the ocean tides can be represented as a graph of a sinusoidal function. At an ocean shipping yard, the water has a maximum depth of 4.2 m above the mean level. Let the vertical axis represent the height above or below the mean level and the horizontal axis represent the cycles (in rads).

   c. The pedals on a bicycle are mounted on a bracket that has the centre 30 cm above the ground. Each pedal is 15 cm from the centre. The pedal starts at its bottom position. Let the vertical axis represent the height of the pedal above the ground and the horizontal axis the angle of rotation (in rads).

d. The times of sunset in an Alberta city vary during the year. On March 21, the sun sets at approximately 1900h; on June 21, the sun sets at approximately 2200h; on September 21, it sets at approximately 1900h, and on December 21, it sets at approximately 1600h. Draw a graph showing at least two more complete rotations. Let the vertical axis represent the time of day and the horizontal axis represent the cycles (in rads).

6. Write the equation of a function for the path of the Ferris wheel in Example 1. Use 30 s as the period of the function instead of 2π.

EXERCISE 7b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 261 (Questions 1, 2)
Mathematics: Principles and Process 12: p. 194 (Questions 10-13), p. 205 (Questions 16-19)
Holtmath 12: p. 306 (Questions 2, 3, 4a, 5a, 5b, 6a, 6b)

Solving a Problem Based on the Graph and Function

Challenge yourself

A Ferris wheel has a radius of 20 m and makes one rotation every 40 s. The lowest point on the Ferris wheel is 1 m above the ground. How high above the ground is a rider after $3\frac{1}{2}$ rotations of the wheel?

MATH FACTOR

Watch Video Segment 7c.

Example 1

Use the data from 7b, Example 1, where one rotation takes 30 s, to determine how high a passenger will be off the ground after 105 s.

Solution

In order to solve this problem, convert the variable of time into radians. Since the Ferris wheel rotates once every 30 s, then 30 s is equivalent to a rotation of 2π rad. Use a ratio to determine how many radians are in a rotation of 105 s duration.

$$\frac{2\pi}{30} = \frac{x}{105}$$
$$210\pi = 30x$$
$$x = \frac{210\pi}{30}$$
$$= 7\pi$$

Use the equation of the function $y = 15\cos\left(\theta - \pi\right) + 16$ and substitute in a value of 7π for θ.

$$y = 15\cos\left(7\pi - \pi\right) + 16$$
$$y = 15\cos\left(6\pi\right) + 16$$
$$y = 15(1) + 16$$
$$y = 31$$

The passenger will be 31 m off the ground after 105 s.

Note: You can also use the sine function.

TEXT

Additional Examples
Mathematics 12: pp. 258-260 (Question 2c)

Exercises

7. A Ferris wheel has a diameter of 40 m and rotates once every 24 s. The passengers get on at 0.5 m above ground. How high will a person be off the ground after 1 min 36 s?

8. In question 5b, find the height of the tide after 62 h if it takes 12.4 h for the height of the tide to repeat itself.

9. In question 5c, the pedal makes 15 rotations per minute. Find the height of the pedal after 45 s.

10. Using data from question 5d, at approximately what time do you expect the sun to set on May 4 (approximately halfway between April 21 and May 21)?

EXERCISE 7c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: pp. 261-263 (Questions 3, 4, 6, 7, 10)
Mathematics: Principles and Process 12: p. 234 (Questions 11, 12)
Holtmath 12: p. 306 (Questions 5c, 5d, 6c, 7a, 7b, 7e, 8c)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. There are many real-world phenomena that can be represented by trigonometric functions.

Solutions

EXERCISE 7a

1. Ferris wheels and other circular objects such as ocean tides or hours of daylight are other applications of sinusoidal waves.

2. a. $g(\theta)$ is louder because a higher amplitude = louder sound.

   b. $f(\theta)$ has a higher pitch because a shorter period = a higher pitch.

3. The same result is found using either method. The horizontal axis is at $y = -1$, and the distance between this line and the maximum point of $y = 1$ is 2. This would indicate that the amplitude is 2.

   The maximum point of 1 minus the minimum point of -3 is equal to 4. This answer divided by 2 is equal to 2. This also indicates that the amplitude is 2.

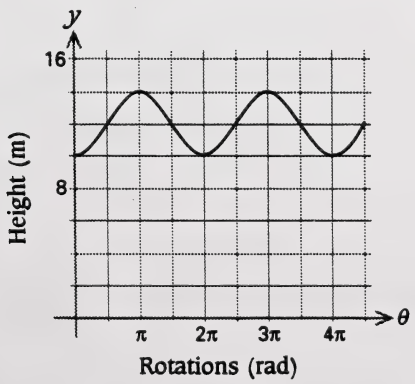
4. a. 2π b. yes

   c. $\frac{\pi}{2}$ units right d. 1 unit down

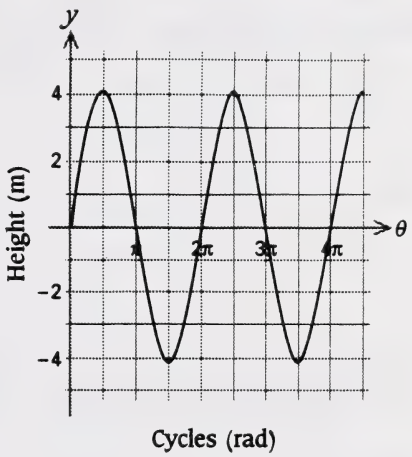
   e. 1 f. $y = 2\sin\left(\theta - \frac{\pi}{2}\right) - 1$

EXERCISE 7b

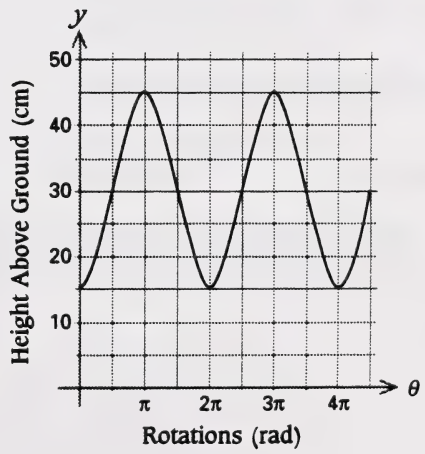
5. a. $y = 2\sin\left(\theta - \frac{\pi}{2}\right) + 12$ or
   $y = 2\cos\left(\theta - \pi\right) + 12$



b. $y = 4.2\sin\theta$ or $y = 4.2\cos\left(\theta - \frac{\pi}{2}\right)$

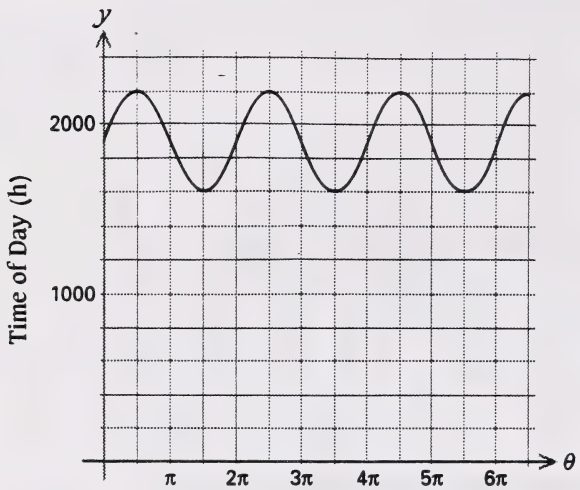


c. $y = 15\sin\left(\theta - \frac{\pi}{2}\right) + 30$ or
$y = 15\cos\left(\theta - \pi\right) + 30$



d. $y = 3\sin\theta + 1900$ or

$y = 3\cos\left(\theta - \frac{\pi}{2}\right) + 1900$



Each Rotation of 2π Equivalent to One Year

6. $\frac{2\pi}{30} = \frac{\pi}{x}$

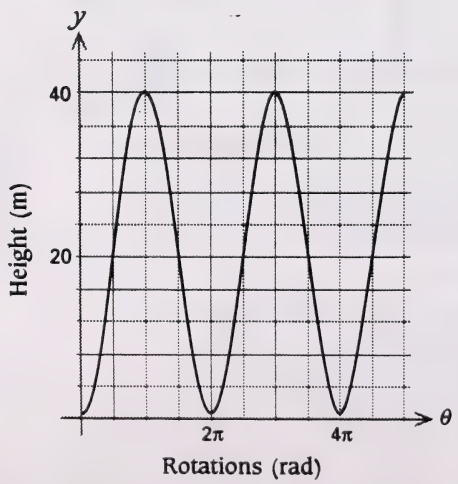
Therefore, $x = 15$ is the phase shift of this function.

$h = 15\cos\left[\frac{2\pi}{30}(t - 15)\right] + 16$

b was calculated using $\frac{2\pi}{b} = 30$ s.

EXERCISE 7c

7. Using the method in 7b, Example 1, draw the graph of the function.



Write an equation that could represent this graph.

$y = 20\sin\left(\theta - \frac{\pi}{2}\right) + 20.5$ or

$y = 20\cos(\theta - \pi) + 20.5$

From the equation, you can determine the value you need.

1 min 36 s is equal to 96 s, which is four times the 24 s required to make a full rotation. Therefore, the equivalent radian measure is $4(2\pi) = 8\pi$ rad.

Substitute this value into the equation to solve for y.

$$\begin{aligned} y &= 20\sin\left(8\pi - \frac{\pi}{2}\right) + 20.5 \\ &= 20\sin\left(\frac{15\pi}{2}\right) + 20.5 \\ &= 20(-1) + 20.5 \\ &= 0.5 \end{aligned}$$

or

$$\begin{aligned} y &= 20\cos(8\pi - \pi) + 20.5 \\ &= 20\cos(7\pi) + 20.5 \\ &= 20(-1) + 20.5 \\ &= 0.5 \end{aligned}$$

This means that a person will be 0.5 m off the ground after 1 min 36 s.

8. Convert 62 h into an equivalent radian measure.

Since 62 h is five times as large as the 12.4 h needed to complete one rotation, the radian measure is $(5)(2\pi) = 10\pi$ rad. Substitute this value into the equation and solve for y.

$$\begin{aligned} y &= 4.2\sin(10\pi) \\ &= 4.2(0) \\ &= 0 \end{aligned}$$

or

$$\begin{aligned} y &= 4.2\cos\left(\theta - \frac{\pi}{2}\right) \\ &= 4.2\cos\left(10\pi - \frac{\pi}{2}\right) \\ &= 4.2\cos\left(\frac{19\pi}{2}\right) \\ &= 4.2(0) \\ &= 0 \end{aligned}$$

This means that after 62 h the tide is at its mean level.

9. If the pedal makes 15 rotations per minute, it means that it makes one rotation every 4 s. If it travels 2π rad every 4 s, then it travels $\frac{\pi}{2}$ rad every second. In 45 s, it will travel $\frac{45\pi}{2}$ rad. Substitute this value into the equation.

$$\begin{aligned} y &= 15\sin\left(\frac{45\pi}{2} - \frac{\pi}{2}\right) + 30 \\ &= 15\sin(22\pi) + 30 \\ &= 15(0) + 30 \\ &= 30 \end{aligned}$$

or

$$\begin{aligned} y &= 15\cos\left(\frac{45\pi}{2} - \pi\right) + 30 \\ &= 15\cos\left(\frac{43\pi}{2}\right) + 30 \\ &= 15(0) + 30 \\ &= 30 \end{aligned}$$

After 45 s, the pedal will be 30 cm off the ground.

10. This graph repeats itself once a year. Each month is $\frac{1}{12}$ of a year, and half way between April 21 and May 21 is 1.5 months away from March 21. This means that it is $\frac{3}{24}$ of a year.

$$\frac{3}{24} \times 2\pi = \frac{\pi}{4}$$

Substitute $\frac{\pi}{4}$ into the function and solve for y.

$$\left(\frac{3}{24}\right)(2\pi) = \frac{\pi}{4}$$

$$\begin{aligned} y &= 3\sin\theta + 19 \\ &= 3\sin\left(\frac{\pi}{4}\right) + 19 \\ &= 3\left(\frac{1}{\sqrt{2}}\right) + 19 \\ &= 21.12 \text{ h or at approximately 2107 h} \end{aligned}$$

or

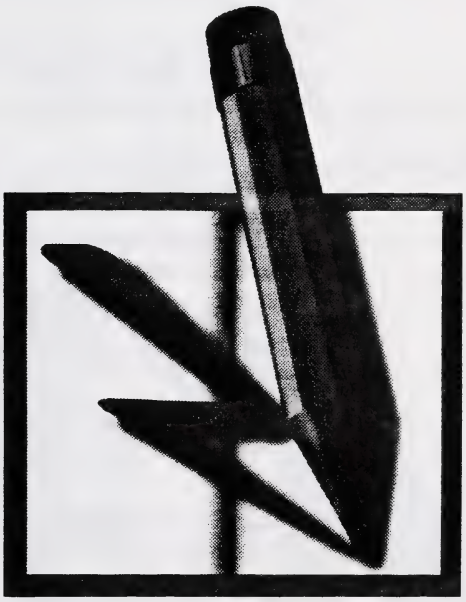
$$\begin{aligned} y &= 3\cos\left(\theta - \frac{\pi}{2}\right) + 19 \\ &= 3\cos\left(\frac{\pi}{4} - \frac{\pi}{2}\right) + 19 \\ &= 3\cos\left(-\frac{\pi}{4}\right) + 19 \\ &= 21.12 \text{ h or at approximately 2107 h} \end{aligned}$$

On this date, the sun will set at approximately 2107 h.

Note: To convert 21.12 to the conventional method of expressing time, multiply 0.12 by 60.

MATH FACTOR

TRIGONOMETRIC AND CIRCULAR FUNCTIONS



Lesson 8: Derivation and Use of Trigonometric Identities

Student Workbook

Produced by

THE EDUCATION STATION

for

Alberta EDUCATION

This guide accompanies **Derivation and Use of Trigonometric Identities** from the Math Factor series.

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Editor: Wilma Gonsalves
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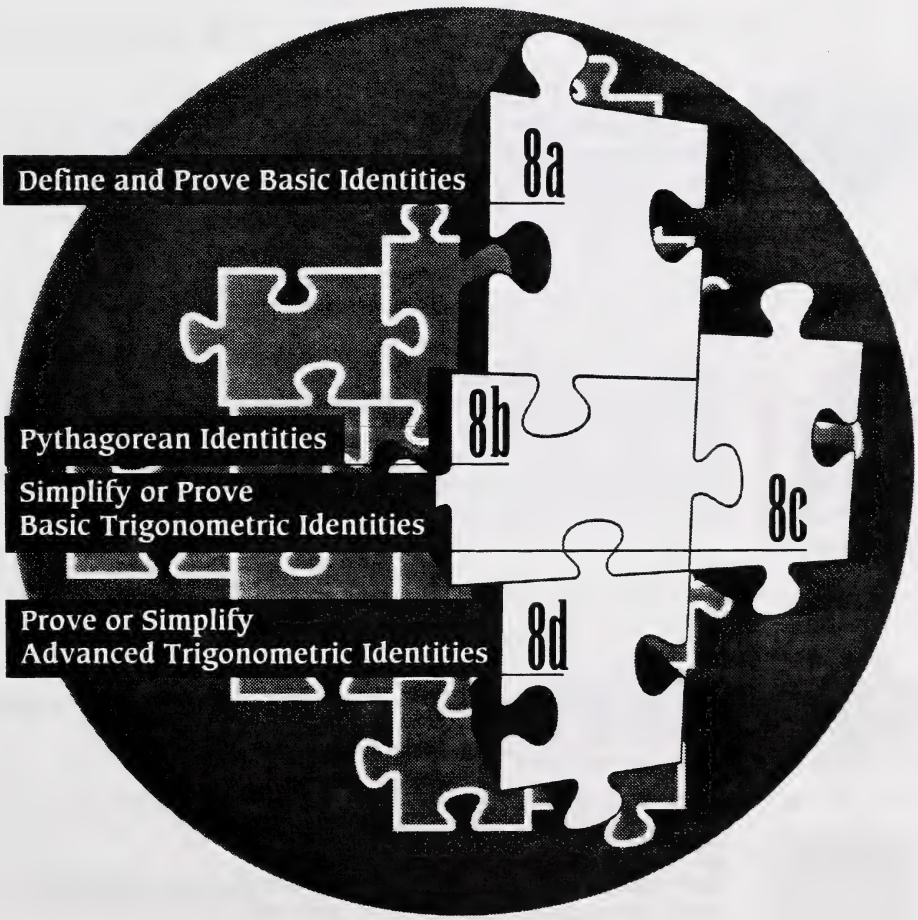
Lesson 8: DERIVATION AND USE OF TRIGONOMETRIC IDENTITIES

LEARNER EXPECTATIONS

By the end of Lesson 8 you should be able to:

- derive the quotient, reciprocal and Pythagorean identities using a logical process
- use the fundamental trigonometric identities to simplify, evaluate and prove basic and advanced trigonometric expressions using identities

This lesson has four parts:



EXPLANATION OF ICONS

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Building Blocks

Do you remember your algebraic skills of factoring, finding a common denominator and making equivalent fractions, multiplying by a conjugate and expanding binomials? You will need these skills when x and y are replaced with trigonometric expressions as you simplify and prove identities.

- Factoring by Difference of Squares

$$1-x^2=(1+x)(1-x)$$

- Common Factoring

$$x^3+xy^2=x\left(x^2+y^2\right)$$

- Trinomial factoring

$$x^2-3x+2=(x-2)(x-1)$$

- Finding a common denominator and combining the two terms

$$1+\frac{y^2}{x^2}=\frac{x^2}{x^2}+\frac{y^2}{x^2}$$
$$=\frac{x^2+y^2}{x^2}$$

$$\frac{1}{1-x}+\frac{1}{1+x}=\frac{1(1+x)+1(1-x)}{(1-x)(1+x)}$$
$$=\frac{1+x+1-x}{1-x^2}$$
$$=\frac{2}{1-x^2}$$

- Multiplying by a Conjugate Binomial

$$\frac{x}{1-x}=\frac{x(1+x)}{(1-x)(1+x)}$$
$$=\frac{x+x^2}{1-x^2}$$

The conjugate binomial of $a+b$ is $a-b$, and of $a-b$ is $a+b$.

- Expanding Binomials

$$(x+y)^2=x^2+2xy+y^2$$

Before attempting the Building Blocks Exercises, carefully study the following examples. You will need to be able to relate the skills you have learned in simplifying algebraic expressions to simplifying trigonometric expressions.

Treat expressions involving trigonometric functions as you would treat variables in algebraic expressions.

- For example, to factor $\sin\theta\cos\theta + 3\cos\theta$, factor it in the same way you would factor the algebraic expression $xy + 3y$.

  $$\sin\theta\cos\theta + 3\cos\theta = \cos\theta(\sin\theta + 3)$$

- An expression such as $\cos^2\theta$ means $(\cos\theta)^2$ and is treated like the algebraic expression x^2. Express $1 - \dfrac{\sin^2\theta}{\cos^2\theta}$ as a single fractional expression .

  $$1 - \frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$$
  $$= \frac{\sin^2\theta - \cos^2\theta}{\cos^2\theta}$$

- Trigonometric expressions may be factored as trinomials. Factor $2\sin^2\theta - 3\sin\theta + 1$ the same way as $2x^2 - 3x + 1$.

  $$2\sin^2\theta - 3\sin\theta + 1 = (2\sin\theta - 1)(\sin\theta - 1)$$

- Expand the binomial $(\sin\theta - 2\cos\theta)^2$.

  $$(\sin\theta - 2\cos\theta)^2 = \sin^2\theta - 4\sin\theta\cos\theta + 4\cos^2\theta 1$$

- Change the form $\dfrac{\sin\theta}{1-\cos\theta}$ by multiplying the numerator and denominator by the conjugate binomial.

  $$\frac{\sin\theta}{1-\cos\theta} = \frac{\sin\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta}$$
  $$= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta}$$

Exercises

1. Factor the following:

   a. $1-\cos^2\theta$

   b. $\cos^3\theta+\cos\theta\sin^2\theta$

   c. $\sec^2\theta-3\sec\theta+2$

2. Find a common denominator and combine the two terms.

   a. $1+\dfrac{\sin^2\theta}{\cos^2\theta}$

   b. $\dfrac{1}{1-\cos\theta}+\dfrac{1}{1+\cos\theta}$

3. Multiply $\frac{\cos\theta}{1-\cos\theta}$ by a conjugate binomial.

4. Expand the binomial $(\cos\theta+\sin\theta)^2$.

5. Simplify $\frac{(\sin\theta-1)}{1-\sin^2\theta}$ by factoring.

BUILDING BLOCKS SOLUTIONS are at the back of this workbook.

Define and Prove Basic Trigonometric Identities

MATH FACTOR

Watch the Introduction and Video Segment 8a.

TEXT

Additional Examples
Mathematics 12: p. 268, p. 269 (bottom), p. 270 (top)
Mathematics: Principles and Process 12: pp. 216, 218
Holtmath 12: p. 316

Exercises

1. An equality statement that holds true when any unrestricted value is substituted for the variable is called an _____.

2. Prove that the two sides of $(x+1)^2 = x^2 + 2x + 1$ are identical.

L.S.	R.S.

3. Verify that $(x+1)^2 = x^2 + 2x + 1$ by substituting $x = -3$.

4. Derive the quotient identities.

   a. Prove that $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

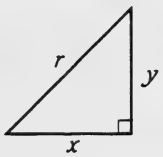
b. Verify that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ by substituting $\theta = \frac{\pi}{6}$ rad.

EXERCISE 8a SOLUTIONS are at the back of this workbook.

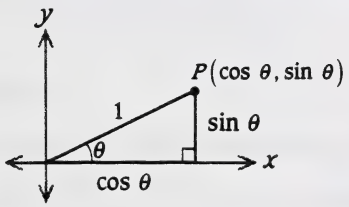
Pythagorean Identities

Exercises

5. a. Using the right triangle given, express the Pythagorean Theorem.



b. Using the right triangle given, express the Pythagorean Identity.



$(\cos\theta)^2 = \cos^2\theta$

$(\sin\theta)^2 = \sin^2\theta$

6. Derive the Pythagorean Identities.

a. Prove that $\cos^2\theta + \sin^2\theta = 1$.

b. Prove that $\cot^2\theta + 1 = \csc^2\theta$.

c. Verify that $1 + \tan^2\theta = \sec^2\theta$ by substituting $\theta = \frac{\pi}{3}$ rad.

7. Complete the following chart.

Quotient Identities	Reciprocal Identities	Pythagorean Identities

EXERCISE 8b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics: Principles and Process 12: p. 217 (Question 1)
Holtmath 12: p. 317 (Questions 1-3)

Simplify or Prove Basic Trigonometric Identities

Challenge yourself

Prove that $\sec^2\theta + \csc^2\theta = (\tan\theta + \cot\theta)^2$.

MATH FACTOR

Watch Video Segment 8c.

General Strategies

- Only work on one side of the table at a time; i.e. work only on the left side or only on the right side.
- Use identities to simplify along with algebraic strategies.

Each of the following examples illustrates different strategies that can be used when proving trigonometric identities.

Example 1

Substituting with Simpler Equivalent Identities

Prove that $\dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta} = \csc\theta$, where $(\sin\theta \neq 0)$.

Solution

	L.S.	R.S.
	$\dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta}$	$\csc\theta$
Pythagorean Identity	$= \dfrac{1}{\sin\theta}$	
Reciprocal Identity	$= \csc\theta$	

L.S. = R.S.

Example 2

Changing All Trigonometric Functions to Sine or Cosine

Express $\dfrac{\tan\theta}{\sec\theta}$ in terms of $\sin\theta$, where $\sec\theta \neq 0$.

Solution

$$\frac{\tan\theta}{\sec\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}$$ Quotient Identity

Reciprocal Identity

$$= \frac{\sin\theta}{\cos\theta} \div \frac{1}{\cos\theta}$$
$$= \frac{\sin\theta}{\cancel{\cos\theta}} \times \frac{\cancel{\cos\theta}}{1}$$
$$= \sin\theta$$

Example 3

Expressing Terms with a Common Denominator

Express $1 + \frac{\sin^2\theta}{\cos^2\theta}$ in terms of $\sec^2\theta$, where $\cos\theta \neq 0$.

Solution

$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}$$
$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$$ Pythagorean Identity
$$= \frac{1}{\cos^2\theta}$$
$$= \sec^2\theta$$

Example 4

Expanding an Expression

Prove that $(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta = 1$.

Solution

	L.S.	R.S.
	$(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta$	1
	$= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta - 2\sin\theta\cos\theta$	
	$= \sin^2\theta + \cos^2\theta$	
Pythagorean Identity	$= 1$	

L.S. = R.S.

TEXT

Additional Examples
Mathematics 12: p. 269 (Example 1), p. 270 (Example 2)
Mathematics: Principles and Process 12: p. 216 (Example), p. 218 (Example 1), p. 219 (Example 2)
Holtmath 12: pp. 317, 318, p. 319 (Steps in Proving Identities)

Exercises

8. Using only $\sin\theta$, $\cos\theta$ and $\tan\theta$, write an equivalent expression for the following:

   a. $1-\cos^2\theta$

   b. $\dfrac{\sec^2\theta}{\csc^2\theta}$, where $\sin\theta$ and $\cos\theta \neq 0$

9. Write the expression $\csc\theta - \cos\theta\cot\theta$ in terms of $\sin\theta$.

10. Prove that $\dfrac{\cot\theta}{\tan\theta} = \dfrac{1-\sin^2\theta}{1-\cos^2\theta}$, where $\cos\theta$ and $\sin\theta \neq 0$.

11. Prove that $\dfrac{\cos\theta - \sin\theta}{\cos\theta} = 1-\tan\theta$, where $\cos\theta \neq 0$, by expressing $1-\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

12. Prove that $\cos^2\theta = (1-\sin\theta)(1+\sin\theta)$.

13. Prove that $\dfrac{\sec^2\theta - 1}{\sec^2\theta}$, where $\cos\theta \neq 0$.

EXERCISE 8c SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 271 (Questions 1-3), p. 272 (Questions 5a, 8a)
Mathematics: Principles and Process 12: p. 217 (Questions 2, 4, 5), p. 219 (Questions 1-4), p. 220 (Questions 6, 7)
Holtmath 12: p. 317 (Questions 4-7, 9-11), p. 319 (Question 2)

Prove or Simplify Advanced Trigonometric Identities

Each of the following examples illustrates a strategy that can be used when proving trigonometric identities.

Example 1

Factoring

Prove that $\frac{\sin^2\theta - 1}{\sin\theta + 1} = \sin\theta - 1$.

Solution

	L.S.	R.S.
Difference of Squares	$\frac{\sin^2\theta - 1}{\sin\theta + 1}$	$\sin\theta - 1$
	$= \frac{(\sin\theta + 1)(\sin\theta - 1)}{(\sin\theta + 1)}$	
	$= \sin\theta - 1$	

L.S. = R.S.

Example 2

Factoring

Prove that $\sin^3\theta + \sin\theta\cos^2\theta = \frac{1}{\csc\theta}$.

Solution

	L.S.	R.S.
Common factor	$\sin^3\theta + \sin\theta\cos^2\theta$	$\frac{1}{\csc\theta}$
Pythagorean Identity	$= \sin\theta(\sin^2\theta + \cos^2\theta)$	
	$= \sin\theta \times 1$	
Reciprocal Identity	$= \frac{1}{\csc\theta}$	

L.S. = R.S.

Example 3

Multiplying by a Conjugate

Prove that $\frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$.

Solution

L.S.	R.S.
$\frac{\sin\theta}{1+\cos\theta}$	$\frac{1-\cos\theta}{\sin\theta}$
$= \frac{\sin\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}$	Multiply by a conjugate.
$= \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}$	
$= \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$	Pythagorean Identity
$= \frac{1-\cos\theta}{\sin\theta}$	

L.S. = R.S.

Additional Examples
Mathematics 12: p. 270 (Example 3)

Exercises

14. Write the expression $\frac{1+\tan\theta}{\sec\theta}$, where $\cos\theta \neq 0$, in terms of $\sin\theta$ and $\cos\theta$.

15. Prove that $\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta} = 2\sec^2\theta$, where $\sin\theta \neq \pm 1$ and $\cos\theta \neq 0$.

16. Prove that $\frac{1+2\sin\theta\cos\theta}{\sin\theta+\cos\theta} = \sin\theta+\cos\theta$, where $\sin\theta,\ \cos\theta \neq 0$.

Hint: Replace 1 with $\sin^2\theta+\cos^2\theta$.

17. Prove that $\frac{\sec\theta+1}{\sec\theta-1}+\frac{\cos\theta+1}{\cos\theta-1}=0$, where $\cos\theta \neq 0, 1$.

18. Redo Example 3 by working with the right side only.

EXERCISE 8d SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 272 (Questions 6a, 7a, 9, 10)
Mathematics: Principles and Process 12: p. 217 (Question 3), p. 220 (Questions 5, 8-10)
Holtmath 12: p. 317 (Question 8), p. 319 (Questions 1, 4, 5), p. 321 (Question 13)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. The trigonometric identities are examined and used in the next lesson. Using these identities is helpful when simplifying and solving trigonometric equations. You should be proficient in recognizing and using the basic identities before continuing on to **Lesson 9: Sum and Difference Identities**.

Solutions

BUILDING BLOCKS

1. a. $(1-\cos\theta)(1+\cos\theta)$ b. $\cos\theta(\cos^2\theta+\sin^2\theta)$ c. $(\sec\theta-2)(\sec\theta-1)$

2. a. $\dfrac{\cos^2\theta}{\cos^2\theta}+\dfrac{\sin^2\theta}{\cos^2\theta}$

$=\dfrac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}$

b. $\dfrac{1(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}+\dfrac{1(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}=\dfrac{1+\cos\theta+1-\cos\theta}{(1-\cos\theta)(1+\cos\theta)}$

$=\dfrac{2}{1-\cos^2\theta}$

3. $\dfrac{\cos\theta(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}=\dfrac{\cos\theta+\cos^2\theta}{1-\cos^2\theta}$

4. $\cos^2\theta+2\cos\theta\sin\theta+\sin^2\theta$ $(\cos\theta)^2=\cos^2\theta$

5. $\dfrac{-1(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}=\dfrac{-1}{1+\sin\theta}$

EXERCISE 8a

1. An equality statement that holds true when any unrestricted value is substituted for the variable is called an **identity**.

2.

L.S.	R.S.
$(x+1)^2$	x^2+2x+1
$=x^2+2x+1$	

L.S. = R.S.

3. $(-3+1)^2=(-3)^2+2(-3)+1$

$(-2)^2=9-6+1$

$4=4$

4. a.

L.S.	R.S.
$\cot\theta$	$\frac{\cos\theta}{\sin\theta}$
$=\frac{x}{y}$	$=\frac{\frac{x}{r}}{\frac{y}{r}}$
	$=\frac{x}{\cancel{r}}\times\frac{\cancel{r}}{y}$
	$=\frac{x}{y}$

L.S. = R.S.

b.

L.S.	R.S.
$\tan\theta$	$\frac{\sin\theta}{\cos\theta}$
$=\tan\frac{\pi}{6}$	$=\frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}}$
$=\frac{1}{\sqrt{3}}$	$=\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$
	$=\frac{1}{\cancel{2}}\times\frac{\cancel{2}}{\sqrt{3}}$
	$=\frac{1}{\sqrt{3}}$

L.S. = R.S.

EXERCISE 8b

5. a. $x^2+y^2=r^2$

b. $(\cos\theta)^2+(\sin\theta)^2=(1)^2$

$\cos^2\theta+\sin^2\theta=1$

6. a.

L.S.	R.S.
$\cos^2\theta+\sin^2\theta$	1
$=\left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2$	
$=\frac{x^2}{r^2}+\frac{y^2}{r^2}$	
$=\frac{x^2+y^2}{r^2}$	
$=\frac{r^2}{r^2}$	
$=1$	

L.S. = R.S.

b.

L.S.	R.S.
$\cot^2\theta+1$	$\csc^2\theta$
$=\left(\frac{x}{y}\right)^2+1$	
$=\frac{x^2}{y^2}+1$	
$=\frac{x^2}{y^2}+\frac{y^2}{y^2}$	
$=\frac{x^2+y^2}{y^2}$	
$=\frac{r^2}{y^2}$	
$=\frac{1}{\sin^2\theta}$	
$=\csc^2\theta$	

L.S. = R.S.

c.

L.S.	R.S.
$1+\tan^2\theta$	$\sec^2\theta$
$=1+\left(\tan\frac{\pi}{3}\right)^2$	$=\left(\sec\frac{\pi}{3}\right)^2$
$=1+\left(\sqrt{3}\right)^2$	$=(2)^2$
$=1+3$	$=4$
$=4$	

L.S. = R.S.

7.

Quotient Identities	Reciprocal Identities	Pythagorean Identities
$\tan\theta=\frac{\sin\theta}{\cos\theta}$ $\cot\theta=\frac{\cos\theta}{\sin\theta}$	$\csc\theta=\frac{1}{\sin\theta}$ $\sec\theta=\frac{1}{\cos\theta}$ $\cot\theta=\frac{1}{\tan\theta}$	$\cos^2\theta+\sin^2\theta=1$ $\tan^2\theta+1=\sec^2\theta$ $\cot^2\theta+1=\csc^2\theta$

Note: These are sample solutions; they are not the only correct method.

EXERCISE 8c

8. a. $1-\cos^2\theta=\sin^2\theta$

b. $\dfrac{\frac{1}{\cos^2\theta}}{\frac{1}{\sin^2\theta}}=\dfrac{1}{\cos^2\theta}+\dfrac{1}{\sin^2\theta}$

$=\dfrac{1}{\cos^2\theta}\times\dfrac{\sin^2\theta}{1}$

$=\dfrac{\sin^2\theta}{\cos^2\theta}$

$=\tan^2\theta$

9. $\csc\theta-\cos\theta\cot\theta=\dfrac{1}{\sin\theta}-\dfrac{\cos\theta\times\cos\theta}{\sin\theta}$

$=\dfrac{1-\cos^2\theta}{\sin\theta}$

$=\dfrac{\sin^2\theta}{\sin\theta}$

$=\sin\theta$

10.

L.S.	R.S.
$\dfrac{\cot\theta}{\tan\theta}$	$\dfrac{1-\sin^2\theta}{1-\cos^2\theta}$
$=\dfrac{\frac{1}{\tan\theta}}{\tan\theta}$	
$=\dfrac{1}{\tan\theta}\div\tan\theta$	
$=\dfrac{1}{\tan\theta}\times\dfrac{1}{\tan\theta}$	
$=\dfrac{1}{\tan^2\theta}$	
$=\cot^2\theta$	
$=\dfrac{\cos^2\theta}{\sin^2\theta}$	
$=\dfrac{1-\sin^2\theta}{1-\cos^2\theta}$	

L.S. = R.S.

11.

L.S.	R.S.
$\dfrac{\cos\theta-\sin\theta}{\cos\theta}$	$1-\tan\theta$
	$=1-\dfrac{\sin\theta}{\cos\theta}$
	$=\dfrac{\cos\theta}{\cos\theta}-\dfrac{\sin\theta}{\cos\theta}$
	$=\dfrac{\cos\theta-\sin\theta}{\cos\theta}$

L.S. = R.S.

12.

L.S.	R.S.
$\cos^2\theta$	$(1-\sin\theta)(1+\sin\theta)$
	$=1-\sin^2\theta$
	$=\cos^2\theta$

L.S. = R.S.

13.

L.S.	R.S.
$\frac{\sec^2\theta - 1}{\sec^2\theta}$	$\sin^2\theta$
$= \frac{\sec^2\theta}{\sec^2\theta} - \frac{1}{\sec^2\theta}$	
$= 1 - \cos^2\theta$	
$= \sin^2\theta$	

Split into two fractions.

Simplify.

Pythagorean Identity

L.S. = R.S.

EXERCISE 8d

14. $\frac{1+\tan\theta}{\sec\theta} = \frac{1+\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}$

$= \frac{\frac{\cos\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}$

$= \frac{\frac{\cos\theta+\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}$

$= \frac{\cos\theta+\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}$

$= \frac{\cos\theta+\sin\theta}{\cancel{\cos\theta}} \times \frac{\cancel{\cos\theta}}{1}$

$= \cos\theta + \sin\theta$

15.

L.S.	R.S.
$\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta}$	$2\sec^2\theta$
$= \frac{1(1-\sin\theta)+1(1+\sin\theta)}{(1-\sin^2\theta)}$	
$= \frac{1-\sin\theta+1+\sin\theta}{\cos^2\theta}$	
$= \frac{2}{\cos^2\theta}$	
$= 2\sec^2\theta$	

L.S. = R.S.

16.

L.S.	R.S.
$\frac{1+2\sin\theta\cos\theta}{\sin\theta+\cos\theta}$	$\sin\theta+\cos\theta$
$=\frac{\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta}{\sin\theta+\cos\theta}$	
$=\frac{(\sin\theta+\cos\theta)^2}{(\sin\theta+\cos\theta)}$	
$=\sin\theta+\cos\theta$	

L.S. = R.S.

17.

Change $\sec\theta$ to $\frac{1}{\cos\theta}$.

Find a common denominator.

$\cos\theta-1=-(1-\cos\theta)$

Find a common denominator.

Simplify.

L.S.	R.S.
$\frac{\sec\theta+1}{\sec\theta-1}+\frac{\cos\theta+1}{\cos\theta-1}$	0
$=\frac{\frac{1}{\cos\theta}+1}{\frac{1}{\cos\theta}-1}+\frac{\cos\theta+1}{\cos\theta-1}$	
$=\frac{\frac{1+\cos\theta}{\cos\theta}}{\frac{1-\cos\theta}{\cos\theta}}+\frac{\cos\theta+1}{\cos\theta-1}$	
$=\frac{1+\cos\theta}{1-\cos\theta}+\left[-\frac{(1+\cos\theta)}{(1-\cos\theta)}\right]$	
$=\frac{1+\cos\theta-(1+\cos\theta)}{(1-\cos\theta)}$	
$=\frac{0}{1-\cos\theta}$	
$=0$	

L.S. = R.S.

18.

L.S.	R.S.
$\frac{\sin\theta}{1+\cos\theta}$	$\frac{1+\cos\theta}{\sin\theta}$
	$=\frac{(1-\cos\theta)(1+\cos\theta)}{\sin\theta(1+\cos\theta)}$
	$=\frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)}$
	$=\frac{\sin^2\theta}{\sin\theta(1+\cos\theta)}$
	$=\frac{\sin\theta}{1+\cos\theta}$

L.S. = R.S.

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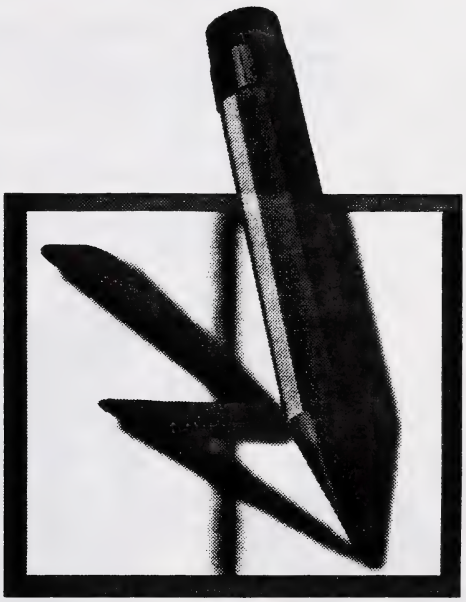
MATH FACTOR

TRIGONOMETRIC AND CIRCULAR FUNCTIONS



Lesson 9:
Sum and Difference Identities

Student Workbook

Produced by

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for

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This guide accompanies **Sum and Difference Identities** from the Math Factor series.

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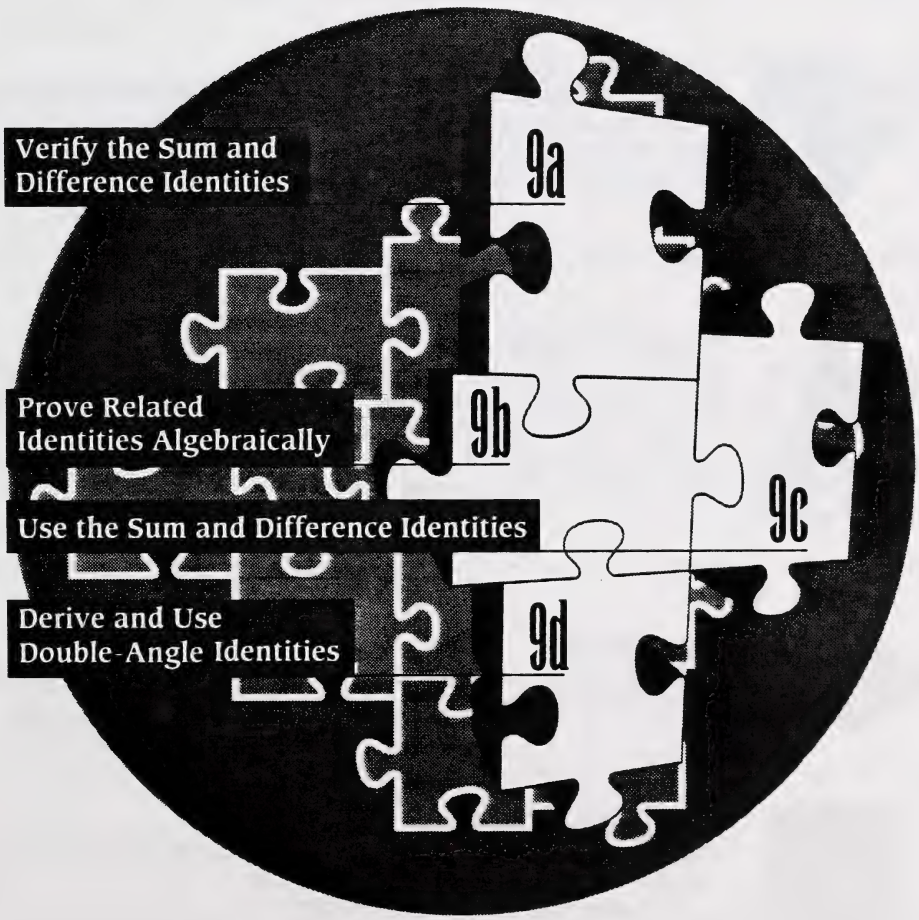
Lesson 9: SUM AND DIFFERENCE IDENTITIES

LEARNER EXPECTATIONS

By the end of Lesson 9 you should be able to:

- verify the sum and difference identities by substitution and graphing
- prove related identities algebraically
- use the sum and difference identities to evaluate, simplify and prove trigonometric statements

This lesson has four parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Verify the Sum and Difference Identities

MATH FACTOR

Watch the Introduction and Video Segment 9a.

Previously you have learned that $3(x + y) = 3x + 3y$. Some people think that $\sin(x + y) = \sin x + \sin y$.

Example 1

Show that $\sin(x + y) \neq \sin x + \sin y$. Suppose $x = \frac{\pi}{3}$ and $y = \frac{\pi}{6}$.

Solution

L.S.	R.S.
$\sin(x + y)$	$\sin x + \sin y$
$= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right)$	$= \sin\frac{\pi}{3} + \sin\frac{\pi}{6}$
$= \sin\left(\frac{\pi}{2}\right)$	$= \frac{\sqrt{3}}{2} + \frac{1}{2}$
$= 1$	$= \frac{\sqrt{3}+1}{2}$
	$\doteq 1.37$

L.S. $\neq$ R.S.

Therefore, $\sin(x + y) \neq \sin x + \sin y$.

TEXT

Additional Examples
Mathematics 12: p. 279, p. 281 (Example 1)
Holtmath 12: p. 322 (Example 1)

Exercises

1. Show that $\cos(a-b) \neq \cos a - \cos b$ using $a = \frac{\pi}{2}$ and $b = \frac{\pi}{4}$.

2. Write the formula for each sum or difference identity.

   a. $\sin(a+b)$

   b. $\sin(a-b)$

   c. $\cos(a+b)$

   d. $\cos(a-b)$

3. Verify that $\sin(a-b) = \sin a \cos b - \cos a \sin b$ using $a = 82°$ and $b = 18°$.

4. Verify that $\sin(a+b) = \sin a \cos b + \cos a \sin b$ using $a = 2.1$ rad and $b = 0.4$ rad.

EXERCISE 9a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 283 (Question 1)
Mathematics: Principles and Process 12: p. 226 (Question 4)
Holtmath 12: p. 324 (Questions 1, 4)

Prove Related Identities Algebraically

Challenge yourself

Prove $\sin(\pi - \theta) = \sin\theta$.

Example 1

Prove that $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$.

Solution

L.S.	R.S.
$\sin\left(\frac{\pi}{2} - \theta\right)$	$\cos\theta$
$= \sin\frac{\pi}{2}\cos\theta - \cos\frac{\pi}{2}\sin\theta$	
$= (1)(\cos\theta) - (0)(\sin\theta)$	
$= \cos\theta$	

L.S. = R.S.

Example 2

Prove that $\sin(\pi + \theta) = -\sin\theta$.

Solution

L.S.	R.S.
$\sin(\pi + \theta)$	$-\sin\theta$
$= \sin\pi\cos\theta + \cos\pi\sin\theta$	
$= (0)(\cos\theta) + (-1)(\sin\theta)$	
$= -\sin\theta$	

L.S. = R.S.

Additional Examples
Mathematics 12: pp. 274-276, p. 277 (Examples 1-3), p. 282 (Example 3)
Mathematics: Principles and Process 12: p. 225 (Example 1), p. 221 (Example 1), p. 222 (Example 2)
Holtmath 12: p. 320, p. 321 (Examples 1, 2)

Exercises

5. Prove the following identities.

   a. $\csc(-\theta) = -\csc\theta$

   b. $\tan(-\theta) = -\tan\theta$

6. Prove the following identities.

   a. $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$

   b. $\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$

7. Simplify the following:

   a. $\cos(\pi + \theta)$

b. $\tan\left(\pi - \theta\right)$

8. a. Prove that $\csc\left(\pi - \theta\right) = \csc\ \theta$.

b. Compare the proof in question 8a to $\sin\left(\pi - \theta\right) = \sin\ \theta$. What pattern do you see for the reciprocal identities?

9. Prove that $\cos\left(\frac{3\pi}{2} + \theta\right) = -\cos\left(\frac{\pi}{2} + \theta\right)$.

EXERCISE 9b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 277 (Questions 1, 2), p. 278 (Questions 3-6, 7a), p. 284 (Questions 8, 9)
Mathematics: Principles and Process 12: pp. 222-223 (Questions 1-6)
Holtmath 12: p. 321 (Questions 5, 6, 8-12)

Use Sum and Difference Identities

Challenge yourself

If $\sin(x+y)=0.6241$ and $\sin(x-y)=0.3201$, find the value of $\sin x \cos y$.

MATH FACTOR

Watch Video Segment 9c.

Example 1

Find an exact value for $\cos \frac{\pi}{12}$.

Solution

$$\cos \frac{\pi}{12} = \cos\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right)$$

$$= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{1}{2}\times\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$$

$$= \frac{\left(1+\sqrt{3}\right)\times\left(\sqrt{2}\right)}{\left(2\sqrt{2}\right)\times\left(\sqrt{2}\right)}$$

Rationalize the denominator.

$$= \frac{\sqrt{2}+\sqrt{6}}{4}$$

Example 2

Find the exact value of $\sin 75^\circ \cos 15^\circ + \cos 75^\circ \sin 15^\circ$.

Solution

$$\sin 75^\circ \cos 15^\circ + \cos 75^\circ \sin 15^\circ = \sin\left(75^\circ + 15^\circ\right)$$

$$= \sin 90^\circ$$

$$= 1$$

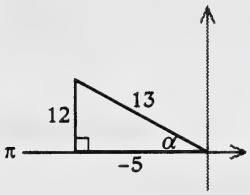
Example 3

If $\cos\alpha = -\frac{5}{13}$ and $\cos\beta = -\frac{3}{5}$, where $\frac{\pi}{2} < \alpha < \pi$ and $\pi < \beta < \frac{3\pi}{2}$, then calculate $\cos(\alpha+\beta) - \sin(\alpha-\beta)$.

Solution

$\cos\alpha = -\frac{5}{13}$; $\sin\alpha = \frac{12}{13}$

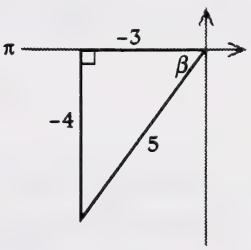
To find the value for $\sin\alpha$, use the Pythagorean Theorem.



$$x^2 + y^2 = r^2$$
$$(-5)^2 + y^2 = (13)^2$$
$$y^2 = 144$$
$$y = 12$$

$\cos\beta = -\frac{3}{5}$; $\sin\beta = -\frac{4}{5}$

To find the value for $\sin\beta$, use the Pythagorean Theorem.



$$x^2 + y^2 = r^2$$
$$(-3)^2 + y^2 = (5)^2$$
$$y^2 = 16$$
$$y = -4$$

$$\cos(\alpha+\beta) - \sin(\alpha-\beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) - (\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$
$$= \left(\frac{-5}{13}\times\frac{-3}{5} - \frac{12}{13}\times\frac{-4}{5}\right) - \left(\frac{12}{13}\times\frac{-3}{5} - \frac{-5}{13}\times\frac{-4}{5}\right)$$
$$= \left(\frac{15}{65} + \frac{48}{65}\right) - \left(\frac{-36}{65} - \frac{20}{65}\right)$$
$$= \frac{63}{65} - \left(\frac{-56}{65}\right)$$
$$= \frac{119}{65}$$

TEXT

Additional Examples
Mathematics 12: p. 282 (Examples 2, 4)
Mathematics Principles and Process 12: p. 225 (Example 2)
Holtmath 12: p. 323 (Example 2)

Exercises

10. Using exact values, simplify the expression $\sin(\theta+60^\circ) - \cos(\theta+30^\circ)$.

11. If $\tan A = -2$, where $\frac{\pi}{2} < A < \pi$, and $\sin B = \frac{3}{5}$, where B is an acute angle, then find $\sin(A+B)$.

12. Determine the exact value of the expression $\cos\frac{7\pi}{12}\cos\frac{5\pi}{12} - \sin\frac{7\pi}{12}\sin\frac{5\pi}{12}$.

13. If $\cos\left(\frac{3\pi}{2} - \theta\right) = -\frac{8}{17}$, then find the value of $\sin\theta$.

14. If $\cos(x+y) = 0.7547$ and $\cos(x-y) = 0.9703$, then find the value of $\cos x \cos y$.

EXERCISE 9c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 283 (Questions 1-6), p. 284 (Questions 10-15)
Mathematics: Principles and Process 12: p. 225 (Question 1), p. 226 (Questions 2-7)
Holtmath 12: p. 324 (Questions 6-10, 12)

Derive and Use Double-Angle Identities

Watch Video Segment 9d.

Additional Examples
Mathematics 12: p. 285 (Examples 1, 2), p. 286 (Example 3), p. 287 (Example 4)
Mathematics: Principles and Process 12: p. 227 (Example 1), p. 228 (Example 2)
Holtmath 12: p. 325 (Examples 1-3)

Exercises

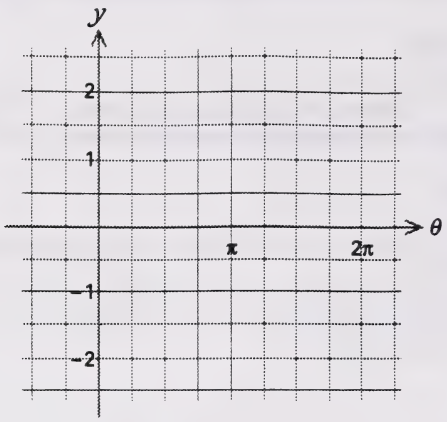
15. Use the Pythagorean Identities $\sin^2\theta = \left(1 - \cos^2\theta\right)$ and $\cos^2\theta = \left(1 - \sin^2\theta\right)$ and the identity for $\cos(0 + \theta)$ to rewrite the following:

    a. $\cos\left(2\theta\right)$ in terms of $\sin\theta$

    b. $\cos\left(2\theta\right)$ in terms of $\cos\theta$

16. Prove that $1 + \cos\left(2\theta\right) = \cot\theta \sin 2\theta$.

17. a. By sketching the graphs of $f(\theta) = \sin 2\theta$ and $g(\theta) = 2\sin\theta$ on the axes, show that $\sin 2\theta \neq 2\sin\theta$.



b. When does $\sin 2\theta = 2\sin\theta$, where $0 \leq \theta \leq 2\pi$?

18. If $\tan\beta = -\frac{8}{15}$ and $\frac{\pi}{2} < \beta < \pi$, then find the value of $\cos(2\beta)$.

19. Prove that $\cos\theta + \sin\theta\tan\theta = 2\sin\theta\csc 2\theta$.

EXERCISE 9d SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics: p. 287 (Questions 1-6), p. 288 (Questions 7-13, 15)
Mathematics: Principles and Process 12: p. 228 (Questions 1-4), p. 229 (Questions 6, 7)
Holtmath 12: p. 324 (Questions 13-15), p. 326 (Questions 1-6)

WHAT HAVE YOU LEARNED?

Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. Your ability to recognize and use the sum and difference identities will enable you to understand and solve complex trigonometric equations. Ensure that you are proficient with these before continuing on to **Lesson 10: Solving Complex First- and Second-Degree Equations and Relationships Between Roots**.

Solutions

EXERCISE 9a

1.

L.S.	R.S.
$\cos(a-b)$	$\cos a - \cos b$
$= \cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$	$= \cos\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{4}\right)$
$= \cos\left(\frac{\pi}{4}\right)$	$= 0 - \frac{1}{\sqrt{2}}$
$= \frac{1}{\sqrt{2}}$	$= -\frac{1}{\sqrt{2}}$

L.S. $\neq$ R.S.

2. a. $\sin(a+b) = \sin a \cos b + \cos a \sin b$

   b. $\sin(a-b) = \sin a \cos b - \cos a \sin b$

   c. $\cos(a+b) = \cos a \cos b - \sin a \sin b$

   d. $\cos(a-b) = \cos a \cos b + \sin a \sin b$

3.

L.S.	R.S.
$\sin(82^\circ - 18^\circ)$	$\sin 82^\circ \cos 18^\circ - \cos 82^\circ \sin 18^\circ$
$= \sin(64^\circ)$	$= (0.9903)(0.9511) - (0.1392)(0.3090)$
$= 0.8988$	$= 0.8988$

degree mode on your calculator

L.S. = R.S.

4.

L.S.	R.S.
$\sin(2.1 + 0.4)$	$\sin 2.1 \cos 0.4 + \cos 2.1 \sin 0.4$
$= \sin(2.5)$	$= (0.8632)(0.9211) + (-0.5048)(0.3894)$
$= 0.5985$	$= 0.5985$

radian mode on your calculator

L.S. = R.S.

EXERCISE 9b

5. a.

L.S.	R.S.
$\csc(-\theta)$	$-\csc\theta$
$=\dfrac{1}{\sin(0-\theta)}$	
$=\dfrac{1}{\sin 0\cos\theta-\cos 0\sin\theta}$	
$=\dfrac{1}{-\sin\theta}$	
$=-\csc\theta$	

L.S. = R.S.

b.

L.S.	R.S.
$\tan(-\theta)$	$-\tan\theta$
$=\dfrac{\sin(0-\theta)}{\cos(0-\theta)}$	
$=\dfrac{-\sin\theta}{\cos 0\cos\theta+\sin 0\sin\theta}$	
$=\dfrac{-\sin\theta}{\cos\theta}$	
$=-\tan\theta$	

L.S. = R.S.

6. a.

L.S.	R.S.
$\sin\left(\frac{\pi}{2}+\theta\right)$	$\cos\theta$
$=\sin\frac{\pi}{2}\cos\theta+\cos\frac{\pi}{2}\sin\theta$	
$=\cos\theta+0$	
$=\cos\theta$	

L.S. = R.S.

b.

L.S.	R.S.
$\tan\left(\frac{\pi}{2}+\theta\right)$	$-\cot\theta$
$=\dfrac{\sin\left(\frac{\pi}{2}+\theta\right)}{\cos\left(\frac{\pi}{2}+\theta\right)}$	
$=\dfrac{\cos\theta}{\cos\frac{\pi}{2}\cos\theta-\sin\frac{\pi}{2}\sin\theta}$	
$=\dfrac{\cos\theta}{-\sin\theta}$	
$=-\cot\theta$	

L.S. = R.S.

7. a. $\cos(\pi+\theta)=\cos\pi\cos\theta-\sin\pi\sin\theta$
$=-\cos\theta$

b. $\tan(\pi-\theta)=\dfrac{\sin(\pi-\theta)}{\cos(\pi-\theta)}$
$=\dfrac{\sin\pi\cos\theta-\cos\pi\sin\theta}{\cos\pi\cos\theta+\sin\pi\sin\theta}$
$=\dfrac{0+\sin\theta}{-\cos\theta+0}$
$=-\tan\theta$

8. a.

L.S.	R.S.
$\csc(\pi-\theta)$	$\csc\theta$
$=\dfrac{1}{\sin(\pi-\theta)}$	
$=\dfrac{1}{\sin\pi\cos\theta-\cos\pi\sin\theta}$	
$=\dfrac{1}{\sin\theta}$	
$=\csc\theta$	

L.S. = R.S.

b. Since $\sin(\pi-\theta)=\sin\theta$ and $\csc(\pi-\theta)=\csc\theta$, the reciprocal identities appear to have the same sign as the primary ratios. Both sine and its reciprocal, cosecant, are positive in quadrant 2.

9.

L.S.	R.S.
$\cos\left(\frac{3\pi}{2}+\theta\right)$	$-\cos\left(\frac{\pi}{2}+\theta\right)$
$=\cos\frac{3\pi}{2}\cos\theta-\sin\frac{3\pi}{2}\sin\theta$	$=-\left[\cos\frac{\pi}{2}\cos\theta-\sin\frac{\pi}{2}\sin\theta\right]$
$=(0)\cos\theta-(-1)(\sin\theta)$	$=-\left[(0)\cos\theta-(1)\sin\theta\right]$
$=\sin\theta$	$=-[-\sin\theta]$
	$=\sin\theta$

L.S. = R.S.

EXERCISE 9c

10. $\sin(\theta+60^\circ)-\cos(\theta+30^\circ)=[\sin\theta\cos 60^\circ+\cos\theta\sin 60^\circ]-[\cos\theta\cos 30^\circ-\sin\theta\sin 30^\circ]$

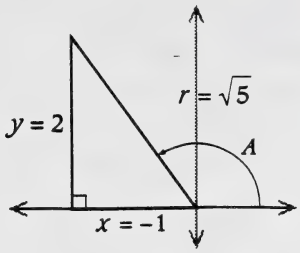
$$=\left[\sin\theta\left(\frac{1}{2}\right)+\cos\theta\left(\frac{\sqrt{3}}{2}\right)\right]-\left[\cos\theta\left(\frac{\sqrt{3}}{2}\right)-\sin\theta\left(\frac{1}{2}\right)\right]$$

$$=\frac{1}{2}\sin\theta+\frac{\sqrt{3}}{2}\cos\theta-\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta$$

$$=\sin\theta$$

11. $\tan A = -2,\ \frac{\pi}{2} < A < \pi$

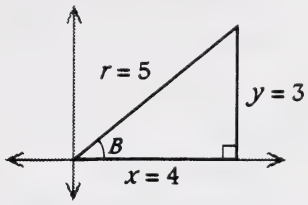
Therefore, A is in quadrant 2.



$$x^2 + y^2 = r^2$$
$$(-1)^2 + (2)^2 = r^2$$
$$r^2 = 5$$
$$r = \sqrt{5}$$

$\sin B = \frac{3}{5}$

Therefore, B is in quadrant 1.



$$x^2 + y^2 = r^2$$
$$x^2 + (3)^2 = (5)^2$$
$$x^2 = 16$$
$$x = 4$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$= \frac{2}{\sqrt{5}} \times \frac{4}{5} + \frac{-1}{\sqrt{5}} \times \frac{3}{5}$$
$$= \frac{8}{5\sqrt{5}} - \frac{3}{5\sqrt{5}}$$
$$= \frac{5}{5\sqrt{5}}$$
$$= \frac{1}{\sqrt{5}}$$
$$= \frac{\sqrt{5}}{5}$$

12. $\cos\frac{7\pi}{12}\cos\frac{5\pi}{12} - \sin\frac{7\pi}{12}\cos\frac{5\pi}{12} = \cos\left(\frac{7\pi}{12} + \frac{5\pi}{12}\right)$
$$= \cos(\pi)$$
$$= -1$$

13. $$\cos\left(\frac{3\pi}{2} - \theta\right) = -\frac{8}{17}$$
$$\cos\left(\frac{3\pi}{2}\right)\cos\theta + \sin\left(\frac{3\pi}{2}\right)\sin\theta = -\frac{8}{17}$$
$$(0)(\cos\theta) + (-1)(\sin\theta) = -\frac{8}{17}$$
$$-\sin\theta = -\frac{8}{17}$$
$$\sin\theta = \frac{8}{17}$$

14. $\cos(x+y) = \cos x\cos y - \sin x\sin y = 0.7547$
$\cos(x-y) = \cos x\cos y + \sin x\sin y = 0.9703$

$$2\cos x\cos y = 1.7250$$
$$\cos x\cos y = 0.8625$$

EXERCISE 9d

15. a. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
$= (1 - \sin^2\theta) - \sin^2\theta$
$= 1 - 2\sin^2\theta$

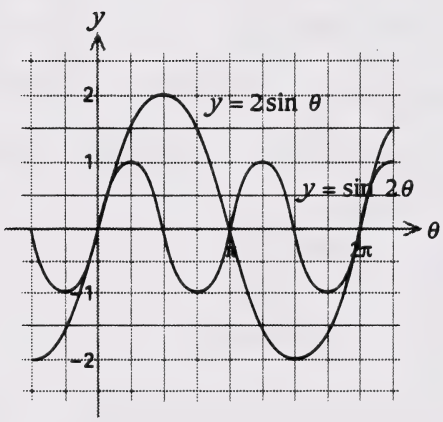
b. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
$= \cos^2\theta - (1 - \cos^2\theta)$
$= \cos^2\theta - 1 + \cos^2\theta$
$= 2\cos^2\theta - 1$

16.

L.S.	R.S.
$1 + \cos 2\theta$	$\cot\theta \sin 2\theta$
	$= \frac{\cos\theta}{\sin\theta} \times \frac{2\sin\theta\cos\theta}{1}$
	$= 2\cos^2\theta$
	$= 1 + 2\cos^2\theta - 1$
	$= 1 + \cos 2\theta$

L.S. = R.S.

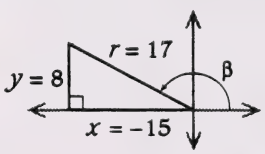
17. a.



b. The values of θ for which $\sin(2\theta)$ and $2\sin\theta$ are equal are $\theta: \{0, \pi, 2\pi\}$.

18. $\tan\beta = -\frac{8}{15}$

Therefore, β is in quadrant 2.



$x^2 + y^2 = r^2$
$(-15)^2 + (8)^2 = r^2$
$289 = r^2$
$r = 17$

$\cos(2\beta) = \cos^2\beta - \sin^2\beta$
$= \left(\frac{-15}{17}\right)^2 - \left(\frac{8}{17}\right)^2$
$= \frac{161}{289}$

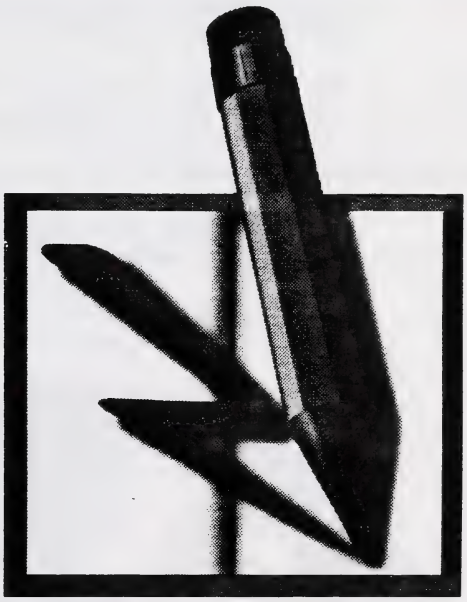
19.

L.S.	R.S.
$\cos\theta + \sin\theta\tan\theta$	$2\sin\theta\csc 2\theta$
$= \cos\theta + \sin\theta \times \frac{\sin\theta}{\cos\theta}$	
$= \cos\theta + \frac{\sin^2\theta}{\cos\theta}$	
$= \frac{\cos^2\theta + \sin^2\theta}{\cos\theta}$	
$= \frac{1}{\cos\theta} \times \frac{2\sin\theta}{2\sin\theta}$	
$= \frac{2\sin\theta}{2\sin\theta\cos\theta}$	
$= \frac{2\sin\theta}{\sin 2\theta}$	
$= 2\sin\theta\csc 2\theta$	

L.S. = R.S.

.2.1997-MDD c2 v10

MATH FACTOR



TRIGONOMETRIC AND CIRCULAR FUNCTIONS

Lesson 10:
Solving Complex First- and Second-Degree Equations and Relationships Between Roots of Trigonometric Equations and Graphs

Student Workbook

Produced by

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This guide accompanies **Solving Complex First- and Second-Degree Equations and Relationships Between Roots of Trigonometric Equations and Graphs** from the Math Factor series.

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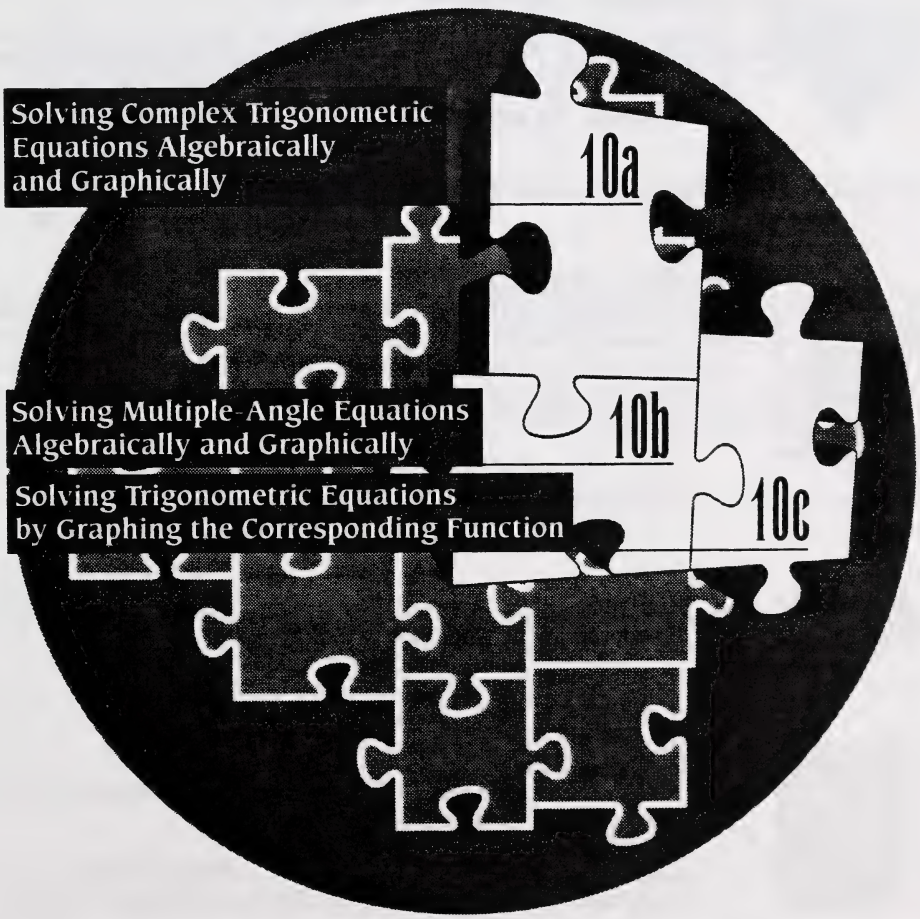
Lesson 10: SOLVING COMPLEX FIRST- AND SECOND-DEGREE EQUATIONS

LEARNER EXPECTATIONS

By the end of Lesson 10 you should be able to

- solve complex first- and second-degree equations – single and multiple angles – algebraically and graphically on either side of the roots
- solve trigonometric equations by graphing the corresponding function (using a graphing calculator or a graphing program)

This lesson has three parts:


Solving Complex Trigonometric Equations Algebraically and Graphically
10a
Solving Multiple-Angle Equations Algebraically and Graphically
10b
Solving Trigonometric Equations by Graphing the Corresponding Function
10c

EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Solving Complex Trigonometric Equations Algebraically and Graphically

To the nearest degree, solve the equation, $2\sec\theta - 1 = 6\csc\theta$, where $0° \le \theta \le 360°$.

Watch the Introduction and Video Segment 10a.

Suggestions for Solving Trigonometric Equations

- Use algebra or trigonometric identities to simplify the equation. A simplified equation contains only a single trigonometric ratio. Find the roots by setting each factor equal to zero and isolate a single trigonometric ratio.

- Address both positive and negative values if you take the square root of both sides of the equation.

- Use your knowledge of special angle triangles and circular functions to find a solution set.

- Ensure that your solutions are within the stated domain.

$0 \le \theta < 2\pi$	: solutions in radians	} solutions possible
$0° \le \theta \le 360°$	: solutions in degrees	} in quadrants 1, 2, 3 or 4
$0 < \theta < \pi$	: solutions only in quadrants 1 and 2	

Example 1

Solve $2\sec^2\theta - \sec\theta - 6$, where $0 \le \theta \le 2\pi$. State your solution set as an exact value or rounded to the nearest tenth.

Solution

$2\sec^2\theta - \sec\theta - 6$, where $0 \le \theta \le 2\pi$ — Solve in radians.

$(2\sec\theta + 3)(\sec\theta - 2) = 0$ — Factor.

$2\sec\theta + 3 = 0$ or $\sec\theta - 2 = 0$ — Set each factor to 0.

$\sec\theta = \frac{-3}{2}$ $\qquad$ $\sec\theta = +2$ — Isolate sec θ.

$\cos\theta = -\frac{2}{3}$ $\qquad$ $\cos\theta = \frac{1}{2}$ — $\cos\theta \neq 0$

θ lies in quadrants 2 and 3. $\qquad$ θ lies in quadrants 1 and 4.

θ ref $\doteq 0.84$ rad $\qquad$ θ ref $= \frac{\pi}{3}$

quadrant 2: $\theta \doteq \pi - 0.84 \doteq 2.3$ $\qquad$ quadrant 1: $\theta = \frac{\pi}{3}$

quadrant 3: $\theta \doteq \pi + 0.84 \doteq 4.0$ $\qquad$ quadrant 4: $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$\theta = \frac{\pi}{3}, 2.3, 4.0, \frac{5\pi}{3}$ — State the solution set.

Example 2

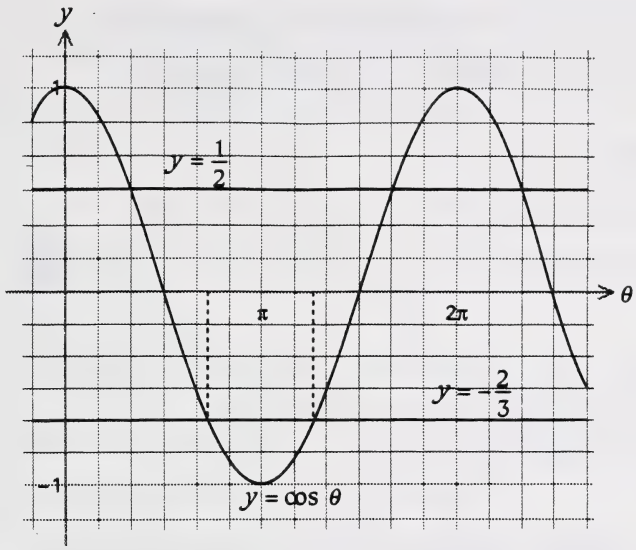
Approximate the solution of $2\sec^2\theta - \sec\theta - 6 = 0$, where $0 \le \theta \le 2\pi$, by drawing the graphs of the function. Verify your solutions by substitution.

Solution

$2\sec^2\theta - \sec\theta - 6 = 0$, where $0 \le \theta \le 2\pi$

$\cos\theta = -\frac{2}{3}$ or $\cos\theta = \frac{1}{2}$

Step 1: Graph $y = \cos\theta$, $y = -\frac{2}{3}$, $y = \frac{1}{2}$.



The estimates of $\theta = 2.3$ and $\theta = 4.0$ were obtained from the graph as follows:

- Each unit on the θ-axis is $\frac{\pi}{6}\left(\frac{\pi}{2} + 3\right)$. This is approximately 0.5236.
- The points of intersection occur at about 4.3 units and 7.6 units.
- Multiply each of these values by 0.5236.
- $4.3 \times 0.5236 \doteq 2.3 \qquad 7.6 \times 0.5236 \doteq 4.0$

$\left(\frac{\pi}{3}, \frac{1}{2}\right)\left(2.3, -\frac{2}{3}\right)\left(4.0, -\frac{2}{3}\right)\left(\frac{5\pi}{3}, \frac{1}{2}\right)$ are the points of intersection.

Step 2: The four solutions are the θ-coordinates of the intersection of the graph of $y = \cos\theta$ and the graphs of $y = \frac{1}{2}$ and $y = \frac{-2}{3}$.

The four solutions (in order) are $\theta = \frac{\pi}{3}, 2.3, 4.0, \frac{5\pi}{3}$. (2.3 and 4.0 are estimates only.)

Step 3: Verify the solutions by substitution in the original equation.

Check $\theta = \frac{\pi}{3}$.

L.S.	R.S.
$2\sec^2\theta - \sec\theta - 6$	0
$= 2\left(\sec^2\frac{\pi}{3}\right) - \sec\frac{\pi}{3} - 6$	
$= 2(+2)^2 - (+2) - 6$	
$= 8 + (-2) - 6$	
$= 0$	

L.S. = R.S.

Check $\theta \doteq 4.0$.

L.S.	R.S.
$2\sec^2\theta - \sec\theta - 6$	0
$= 2(\sec 4.0)^2 - \sec 4.0 - 6$	
$= 2(-1.5)^2 - (-1.5) - 6$	
$= 4.5 + 1.5 - 6$	
$= 0$	

L.S. = R.S.

TEXT

Additional Examples
Mathematics 12: p. 289 (Example 1), p. 290 (Examples 2, 3)
Mathematics: Principles and Process 12: p. 230, p. 232 (Examples 1, 2)
Holtmath 12: p. 327 (Examples 1, 2), p. 328 (Example 3)

Exercises

1. Solve the equation $2\sin^2\theta - 1 = 0$, where $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$.

2. Solve the equation $\sin\theta\tan\theta - \sin\theta = 0$, where $0^\circ \le \theta \le 360^\circ$.

3. Show the algebraic solution to each of the following equations.

   a. $10\sin\theta = 2\csc\theta + 1$, where $0^\circ \le \theta \le 360^\circ$

   b. $4\cos^4\theta - 3\sin^2\theta = -2$, where $\frac{3\pi}{2} \le \theta \le 2\pi$

c. $\sin 2\theta = \sin \theta$, where $0 \leq \theta \leq 2\pi$

EXERCISE 10a SOLUTIONS are at the back of this workbook.

Additional Practice

Mathematics 12: p. 293 (Questions 1, 2, 3a, 3b, 3d, 4c, 4d),
p. 297 (Questions 11a, 11b, 11d), p. 298 (Questions 10a, 10b)
Mathematics: Principles and Process 12: p. 233 (Questions 4-8),
p. 230 (Questions 1-5)
Holtmath 12: p. 329 (Questions 7, 13, 14, 16)

Solving Multiple-Angle Equations Algebraically and Graphically

Challenge yourself

Solve the equation $2\sin\theta\cos\theta = -1$, where $0 \le \theta \le 2\pi$.

MATH FACTOR

Watch Video Segment 10b.

Example

Solve the equation $2\cos^2(2\theta) - 1 = 0$, where $0 \le \theta \le 2\pi$.

Solution

Algebraic Solution

When solving for multiple angles, adjust the domain.

$0 \le \theta < 2\pi$
$\therefore 0 \le 2\theta \le 4\pi$

Let $\alpha = 2\theta$.
$\therefore 0 \le \alpha \le 4\pi$

The equation now becomes as follows:

$$2\cos^2\alpha - 1 = 0, \text{ where } 0 \le \alpha \le 4\pi$$
$$\therefore \cos^2\alpha = \frac{1}{2}$$
$$\cos\alpha = \pm\frac{1}{\sqrt{2}}$$
$$\alpha \text{ ref} = \frac{\pi}{4}$$
$$\alpha = \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}\right\}$$
$$\therefore 2\theta = \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}\right\}$$
$$\therefore \theta = \left\{\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}\right\}$$

Graphical Solution

Graph $y = \cos(\alpha)$, where $0 \le \alpha \le 4\pi$.

$y = \cos(\alpha)$ is
$y = \cos(2\theta)$.

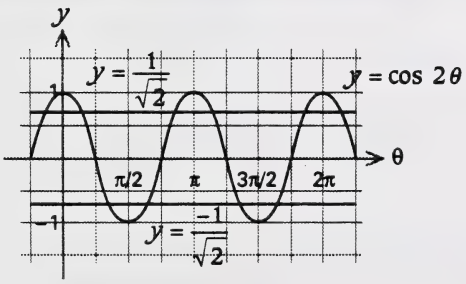
Graph $y = \frac{1}{\sqrt{2}}, y = \frac{-1}{\sqrt{2}}$.

In the video examples, the method used is to let $2\theta = \alpha$ and graph solutions for α for a domain of $0 \le \alpha < 4\pi$. Then divide each solution by 2 to solve for θ for a domain of $0 \le \theta < 2\pi$.

You may find it easier to graph $y = \cos(2\theta)$ for a domain of $0 \le \theta < 2\pi$.

Graph $y = \cos(2\theta)$. The new period is π and for the domain $0 \le \theta < 2\pi$, you will see two complete cycles.

Graph $y = \frac{1}{\sqrt{2}}, y = -\frac{1}{\sqrt{2}}$.



The eight solutions for θ are the θ-coordinates of the intersection points of the graph of $y = \cos 2\theta$, with the horizontal lines $y = \pm\frac{1}{\sqrt{2}}$.

The solutions for θ are as follows:

$$\theta = \left\{\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}\right\}$$

The solutions for θ are, therefore, the α or 2θ solutions divided by 2. Either method is acceptable.

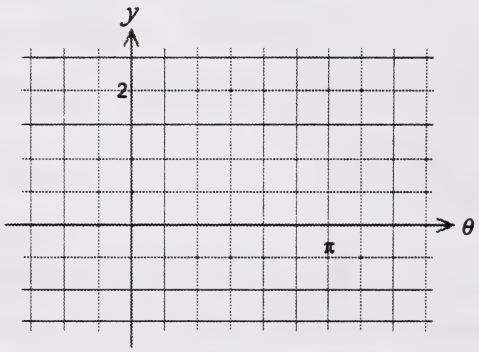
TEXT

Additional Example
Holtmath 12: p. 332 (Example 1)

Exercises

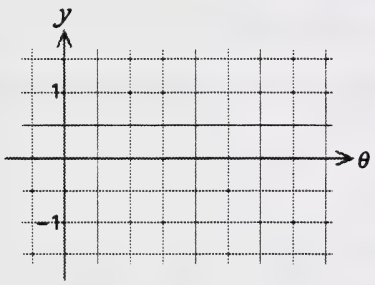
4. a. Solve $2\cos^2(2\theta) - 7\cos(2\theta) = -3$, where $0 \le \theta \le \pi$, algebraically.

b. Solve $2\cos^2(2\theta) - 7\cos(2\theta) = -3$, where $0 \le \theta \le \pi$, graphically.



5. a. Solve $4\sin^2\left(\frac{1}{2}\theta\right) = 3$, where $0° \le \theta \le 360°$, algebraically.

b. Solve $4\sin^2\left(\frac{1}{2}\theta\right) = 3$, where $0° \le \theta \le 360°$, graphically.



6. In a previous lesson, you solved the equation $\sin(2\theta) = 2\sin\theta$, where $0 \le \theta \le 2\pi$, by graphing and found the solution to be $\theta = 0, \pi, 2\pi$. Now solve the equation algebraically to verify these solutions.

EXERCISE 10b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 293 (Question 4b)
Mathematics: Principles and Process 12: p. 231 (Questions 6, 8), p. 233 (Question 10)
Holtmath 12: p. 333 (Questions 2-4)

Solving Trigonometric Equations by Graphing the Corresponding Function

If you have access to a graphing calculator or a computer graphing program, you can graph the equation and find your solutions in the following manner.

Step 1: Set the equation equal to zero.

Step 2: Graph the corresponding function.

Step 3: The x- or θ-intercept(s) within the stated domain are the solution set of the equation.

Example

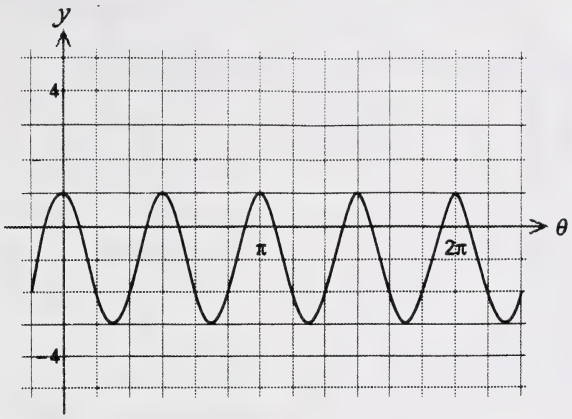
Graph $4\cos^2(2\theta) = 3$, where $0 \le \theta \le 2\pi$.

Step 1: $4\cos^2(2\theta) - 3 = 0$

Step 2: $y = 4\cos^2(2\theta) - 3$

Step 3: x- or θ-intercepts are the zeros of the function and are, therefore, the roots of the equation.

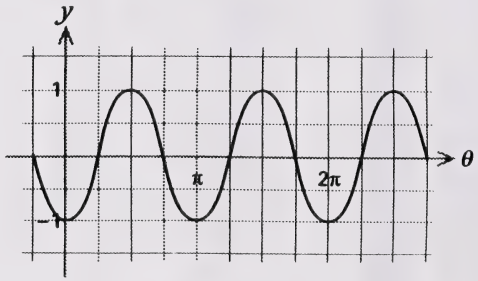
Note: The zeros of a function given by a calculator or a computer are approximations.



$\theta = \{0.26, 1.31, 1.83, 2.88, 3.40, 4.45, 4.97, 6.02\}$

$\theta = \left\{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}$

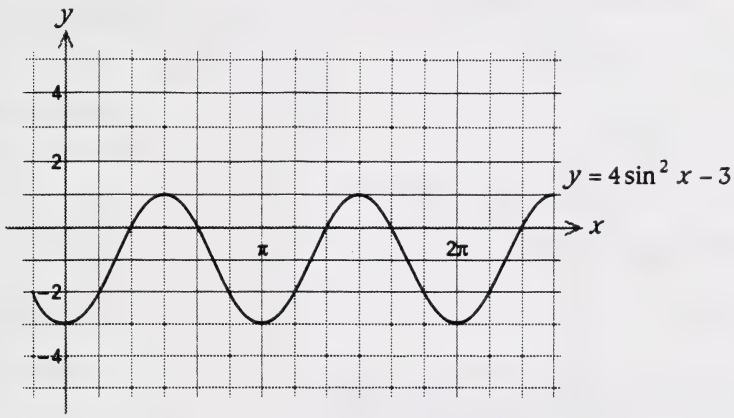
Summary: Solving Trigonometric Equations

Method 1	Method 2	Method 3
Algebraic Method	**Graphing Both Sides of the Roots**	**Graphing the Corresponding Function**
Step 1: Simplify the equation using the rules of algebra and trigonometric identities. A simplified equation contains only a single trigonometric ratio. **Step 2:** Use your knowledge of special angle triangles and circular functions to find a solution set for x within the given domain.	**Step 1:** Simplify the equation. **Step 2:** Graph both sides of the roots of each equation. **Step 3:** The x-coordinates of the point(s) of intersection of the graphs within the domain are the solution set of the equation.	**Step 1:** Set the original equation equal to zero. Let $y = 0$. **Step 2:** Graph the corresponding function and examine the zeros (x-intercepts). **Step 3:** The x-intercept(s) within the domain are the solution set of the equation.
Example $\sin^2 x = \cos^2 x$, where $0 \le x$ $\sin^2 x - \cos^2 x = 0$ $\sin^2 x - (1 - \sin^2 x) = 0$ $2\sin^2 x - 1 = 0$ x ref : $\frac{\pi}{4}$ $\quad \sin^2 x = \frac{1}{2}$ $\sin x = \pm\frac{1}{\sqrt{2}}$ quadrant $1 = \frac{\pi}{4}$ quadrant $2 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ quadrant $3 = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$ quadrant $4 = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$	**Example** $\sin^2 x = \cos^2 x$, where $0 \le x \le 2\pi$ $\sin^2 x - \cos^2 x = 0$ $\sin x = \pm\frac{1}{\sqrt{2}}$ $\sin x = \frac{1}{\sqrt{2}}$ or $\sin x = \frac{-1}{\sqrt{2}}$ = Graph $y = \sin x$, $y = \frac{1}{\sqrt{2}}$, $y = \frac{-1}{\sqrt{2}}$. $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right)\left(\frac{5\pi}{4}, \frac{-1}{\sqrt{2}}\right)\left(\frac{7\pi}{4}, \frac{-1}{\sqrt{2}}\right)$	**Example** $\sin^2 x = \cos^2 x$, where $0 \le x \le 2\pi$ $\sin^2 x - \cos^2 x = 0$ $y = \sin^2 x - \cos^2 x$  $\left(\frac{\pi}{4}, 0\right)\left(\frac{3\pi}{4}, 0\right)\left(\frac{5\pi}{4}, 0\right)\left(\frac{7\pi}{4}, 0\right)$
$x = \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$	$x = \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$	$x = \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

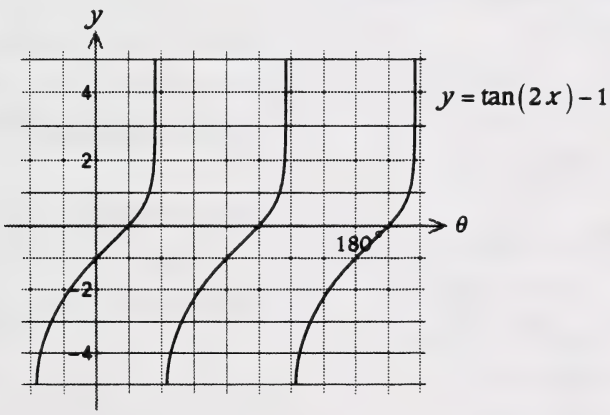
Exercises

7. Use the graphs shown to solve the equations for x.

   a. $4\sin^2 x - 3 = 0$, where $0 \le x \le 2\pi$



   b. $\tan(2x) = 1$, where $0° \le x \le 180°$



EXERCISE 10c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: Principles and Process 12: p. 263 (Questions 1, 2)
Holtmath 12: p. 346 (Question 1)

WHAT HAVE YOU LEARNED?
Return to the beginning of this lesson and check off the Learner Expectations that you have mastered. In order to solve complex trigonometric equations, you must be proficient at applying your knowledge of algebra, special angle triangles and circular functions, trigonometric identities, graphing the primary trigonometric functions and solving equations. The next lesson, **Review**, will assist you in reviewing the basic concepts and applying several concepts to solve more complex questions.

Solutions

EXERCISE 10a

1. $2\sin^2\theta - 1 = 0$, where $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

$$\sin^2\theta = \frac{1}{2}$$

$$\sin\theta = \pm\frac{1}{\sqrt{2}}$$

θ ref $= \frac{\pi}{4}$ θ lies in quadrants 2 and 3.

$\theta: \pi - \frac{\pi}{4} = \frac{3\pi}{4}, \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

$\theta: \left\{\frac{3\pi}{4}, \frac{5\pi}{4}\right\}$

In general, the θ-coordinates of the intersection of the graphs of both sides of the roots of the equation are the solution set for the equation.

2. $\sin\theta\tan\theta - \sin\theta = 0$, where $0 \le \theta \le 360°$

$$\sin\theta(\tan\theta - 1) = 0$$

$\sin\theta = 0$ or $\tan\theta - 1 = 0$

$\tan\theta = 1$

$\theta: \{0°, 45°, 180°, 225°\}$

3. a. $10\sin\theta = 2\csc\theta + 1$, where $0° \le \theta \le 360°$

$$10\sin\theta = \frac{2}{\sin\theta} + 1$$

$$10\sin^2\theta = 2 + \sin\theta$$

$$10\sin^2\theta - \sin\theta - 2 = 0$$

$$(5\sin\theta + 2)(2\sin\theta - 1) = 0$$

$\sin\theta = \frac{-2}{5}$ or $\sin\theta = \frac{1}{2}$

θ ref $= 23.6°$

quadrant 3: $180° + 23.6° \doteq 203.6°$

quadrant 4: $360° - 23.6° \doteq 336.4°$

θ ref $= 30°$

quadrant 1: $30°$

quadrant 2: $180° - 30° = 150°$

$\theta = \{30.0°, 150.0°, 203.6°, 336.4°\}$

b. $4\cos^4\theta - 3\sin^2\theta = -2$, where $\frac{3\pi}{2} \le \theta \le 2\pi$

$$4\cos^4\theta - 3(1 - \cos^2\theta) = -2$$

$$4\cos^4\theta - 3 + 3\cos^2\theta = -2$$

$$4\cos^4\theta + 3\cos^2\theta - 1 = 0$$

$$(4\cos^2\theta - 1)(\cos^2\theta + 1) = 0$$

$4\cos^2\theta - 1 = 0$ or $\cos^2\theta + 1 = 0$

$4\cos^2\theta = 1$ $\cos^2\theta = -1$

$\cos^2\theta = \frac{1}{4}$ There is no solution in the set of real numbers.

$\cos\theta = \pm\frac{1}{2}$ Since $\frac{3\pi}{2} \le \theta \le 2\pi$, θ is in quadrant 4.

θ ref $= \frac{\pi}{3}$

$\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$\theta = \left\{\frac{5\pi}{3}\right\}$

Verify by substitution.

L.S.	R.S.
$4\cos^2\left(\frac{5\pi}{3}\right) - 3\sin^2\left(\frac{\pi}{3}\right)$	-2
$= 4\left(+\frac{1}{2}\right)^4 - 3\left(\frac{\sqrt{3}}{2}\right)^2$	
$= \frac{1}{4} - \frac{9}{4}$	
$= -2$	

L.S. = R.S.

c. $\sin 2\theta = \sin\theta$, where $0 \le \theta \le 2\pi$

$2\sin\theta\cos\theta = \sin\theta$

$2\sin\theta\cos\theta - \sin\theta = 0$

$\sin\theta(2\cos\theta - 1) = 0$

$\sin\theta = 0$ or $2\cos\theta - 1 = 0$

$\cos\theta = \frac{1}{2}$

θ ref $= \frac{\pi}{3}$

quadrant 1: $\frac{\pi}{3}$

quadrant 4: $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$\theta = 0, \pi$ $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$\theta = \left\{0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$

EXERCISE 10b

4. a. $2\cos^2(2\theta) - 7\cos(2\theta) = 3$, where $0 \le \theta \le \pi$

$2\cos^2(2\theta) - 7\cos(2\theta) + 3 = 0$, where $0 \le 2\theta \le 2\pi$

$(2\cos(2\theta) - 1)(\cos(2\theta) - 3) = 0$, where $0 \le 2\theta \le 2\pi$

$2\cos(2\theta) - 1 = 0$ $\cos(2\theta) - 3 = 0$

$2\cos(2\theta) = 1$ $\cos(2\theta) = 3$

$\cos(2\theta) = \frac{1}{2}$ No solutions

$\cos\alpha = \frac{1}{2}$

(2θ) ref $= \frac{\pi}{3}$; Let $\alpha = 2\theta$

α ref $= \frac{\pi}{3}$

quadrant 1: $\frac{\pi}{3}$

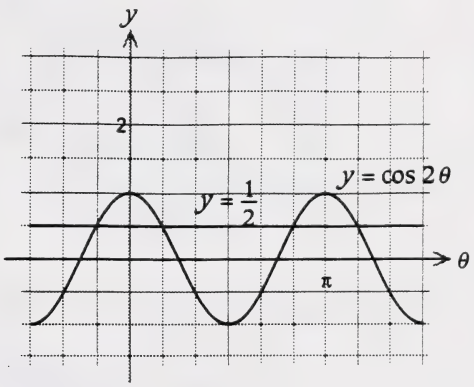
quadrant 4: $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$\alpha\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$ for $0 \le \alpha < 2\pi$

$\frac{\alpha}{2} = \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$ for $0 \le \theta < \pi$ Solution

b. Draw the graph of

$y = \cos(2\theta)$, $y = 3$ and $y = 0.5$.



The points of intersection are

$\left(\frac{\pi}{6}, 0.5\right)\left(\frac{5\pi}{6}, 0.5\right)$.

$\therefore \theta = \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$

5. a. $4\sin^2\left(\frac{1}{2}\theta\right) = 3$, where $0° \le \theta < 360°$

$\sin^2\left(\frac{1}{2}\theta\right) = \frac{3}{4}$, where $0° \le \left(\frac{1}{2}\theta\right) < 180°$

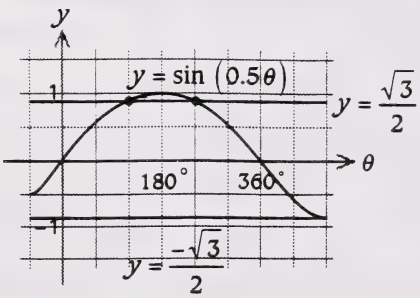
$\sin\left(\frac{1}{2}\theta\right) = \pm\frac{\sqrt{3}}{2}$, where $0° \le \alpha < 180°$

$\left(\frac{1}{2}\theta\right)$ ref $= 60°$; Let $\alpha = \frac{1}{2}\theta$.

$\alpha = \{60°, 120°\}$

$2\alpha = \theta = \{120°, 240°\}$ Solution

b. Graph $y = \sin\left(\frac{1}{2}\theta\right)$, $y = \frac{\sqrt{3}}{2}$, $y = \frac{-\sqrt{3}}{2}$, where $0° \le \theta \le 360°$.



The points of intersection are $\left(120°, \frac{\sqrt{3}}{2}\right)\left(240°, \frac{\sqrt{3}}{2}\right)$.

$\therefore \theta = \{120°, 240°\}$ Solution

6. $\sin(2\theta) = 2\sin\theta$, where $0 \le \theta \le 2\pi$

$2\sin\theta\cos\theta - 2\sin\theta = 0$

$2\sin\theta(\cos\theta - 1) = 0$

$2\sin\theta = 0$ or $\cos\theta - 1 = 0$

$\sin\theta = 0$ $\cos\theta = 1$

$\therefore \theta = \{0, \pi, 2\pi\}$

EXERCISE 10c

7. a. $x = \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$

b. $x = \{22.5°, 112.5°\}$

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MATH FACTOR

TRIGONOMETRIC AND CIRCULAR FUNCTIONS

Lesson 11:
Review

Student Workbook

Produced by

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EDUCATION

This guide accompanies **Review** from the Math Factor series.

Writer: Heather Baile
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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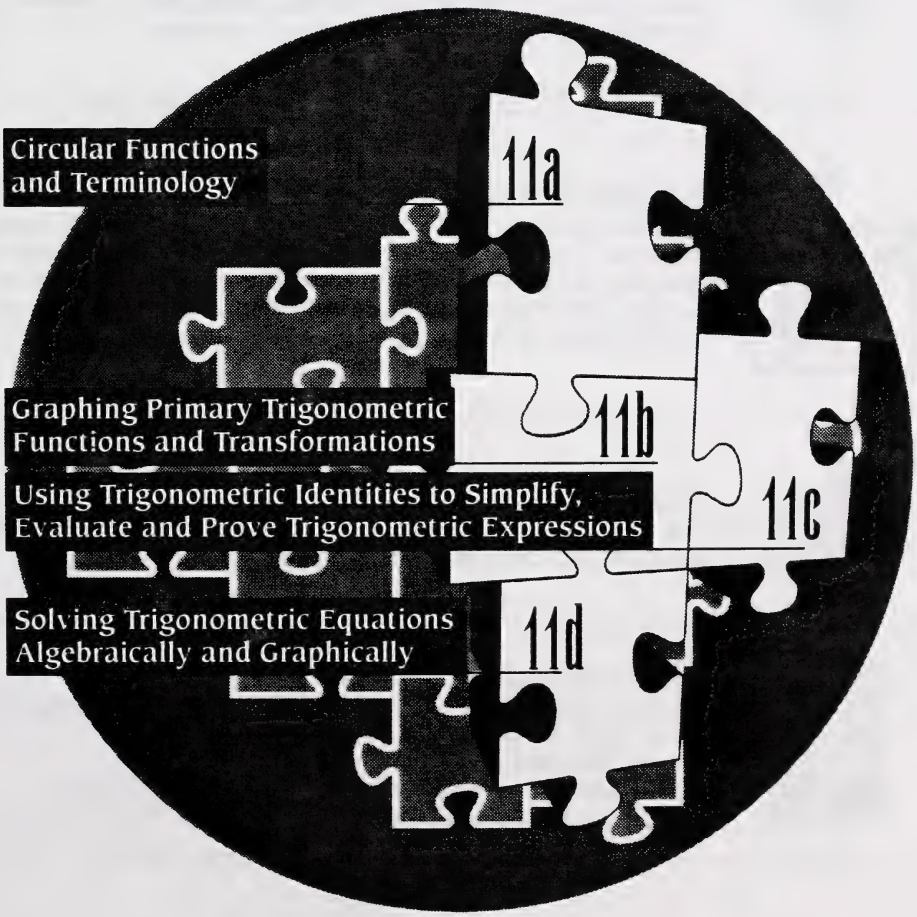
Lesson 11: REVIEW

LEARNER EXPECTATIONS

By the end of Lesson 11 you should be able to:

- demonstrate an understanding of radian measure and utilize both degrees and radians to solve trigonometric and circular function questions, as well as basic trigonometric equations
- graph the primary trigonometric functions and graph transformations of the sine and cosine graphs, as well as develop and use the trigonometric functions in practical problem solving
- use reciprocal, quotient, Pythagorean and the sum/difference identities for proofs and simplification of trigonometric equations
- demonstrate an ability to utilize previous trigonometric concepts to solve complex trigonometric equations and application problems

This lesson has four parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Circular Functions and Terminology

Watch the Introduction and Video Segment 11a.

The material in this video segment is a review of *Trigonometric and Circular Functions*, Lessons 1, 2 and 3.

Exercises

1. In which quadrant does the terminal side of an angle θ in standard position lie if $\tan\theta > 0$ and $\sin\theta < 0$?

2. If θ is an angle in quadrant 3 and $\cos\theta = \frac{-1}{2}$, then calculate the exact value of $\sin\theta\cos\theta + \tan^2\theta$.

3. If $\tan\theta = m$, $\sin\theta < 0$ and $m > 0$, then express the value of $\csc\theta$ in terms of m.

4. Define one radian.

5. Change the following to degrees.

   a. $\frac{3\pi}{4}$ rad = _____

   b. 2.3 rad ≈ _____

6. Change the following to radians.

   a. $270° =$ _____ rad

   b. $320° \doteq$ _____ rad

7. If $\sec\theta = \frac{-5}{3}$, find the following:

   a. $\sin\theta$

   b. the measure of θ in both degrees and radians (to the nearest tenth)

8. Give the exact value of the following:

   a. $\cos\left(\frac{-13\pi}{6}\right)$ _____

   b. $\cot\left(\frac{25\pi}{3}\right)$ _____

9. Find the exact value of $\sin^2\left(\frac{5\pi}{3}\right)\times\cos^2\left(-\frac{\pi}{4}\right)-\cot\left(\frac{7\pi}{4}\right)\times\left(-\sec\frac{\pi}{6}\right)$.

EXERCISE 11a SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 266 (Questions 1-9), p. 298 (Questions 1-5)
Mathematics: Principles and Process 12: p. 212 (Questions 1-5), p. 213 (Questions 1-5), p. 214 (Question 8)
Holtmath 12: p. 311 (Questions 3-8), p. 312 (Questions 4-9), p. 314 (Questions 19-21)

Graphing Primary Trigonometric Functions and Transformations

Watch Video Segment 11b.

The material in this video segment is a review of *Trigonometric and Circular Functions*, Lessons 4, 5, 6 and 7.

Exercises

10. a. What is the range of $f(x) = -7\sin(3x) + 6$?

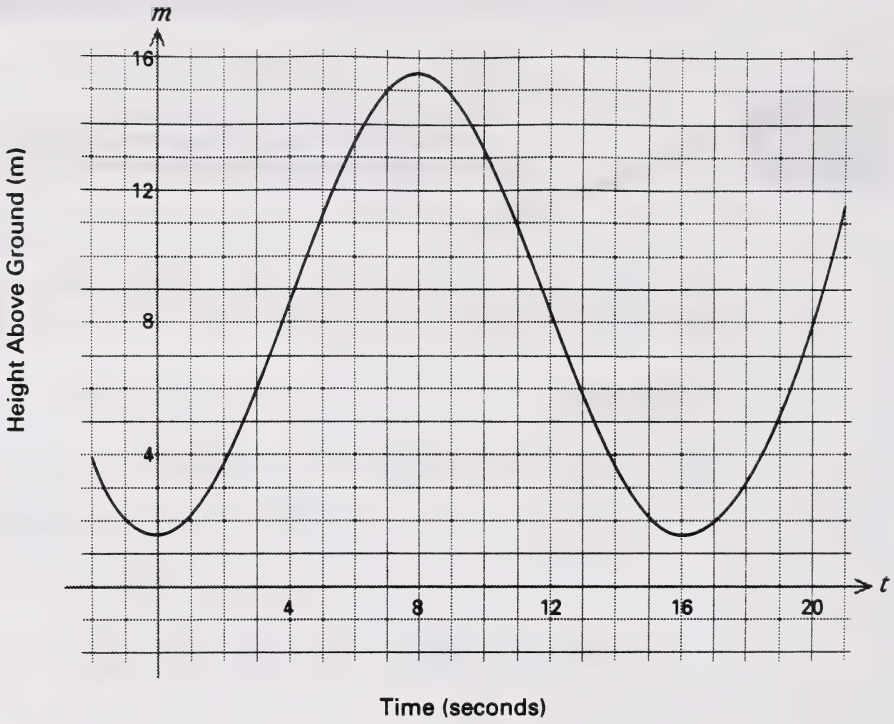
    b. What is the range of $f(x) = -k\cos(bx) + d$?

11. For what value of k is the period of the graph of $y = \frac{1}{3}\cos kx$ equal to $\frac{2\pi}{3}$?

12. Identify the θ-intercepts of $y = -3\cos(3\theta)$ for the domain $0 \le \theta < 2\pi$.

EXCELLENCE

13. The height of a chair on a Ferris wheel at time t seconds is shown on the graph. For example, when $t = 2$ s, the height of the chair above the ground is about 3.7 m. As time passes and the wheel rotates, the height of the chair will rise and fall in the form of a sinusoidal wave. Use the graph to answer the questions that follow.



a. How long does the wheel take to make one complete rotation?

b. Approximate the maximum height the chair will reach.

c. Approximate the minimum height the chair will reach.

d. What is the amplitude of this sinusoidal function?

e. Estimate the height at $t = 10$ s and at $t = 12$ s.

f. Express the exact value of the angular velocity of the chair in radians per second.

g. If the equation of the function is given by $h(t) = a\sin\ b(t + c) + d$, find the values of a, b, c and d.

h. Use the equation found in question 13.g to find the height of a chair after 40 s and again after 64 s.

EXERCISE 11b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice

Mathematics 12: p. 266 (Questions 10-13), p. 298 (Question 6)
Mathematics: Principles and Process 12: p. 212 (Question 6), p. 213 (Questions 7-9), p. 214 (Question 7)
Holtmath 12: p. 311 (Questions 10, 11), p. 312 (Questions 10, 11), p. 314 (Questions 22, 23)

Using Trigonometric Identities to Simplify, Evaluate and Prove Trigonometric Expressions

Watch Video Segment 11c.

The material in this video segment is a review of *Trigonometric and Circular Functions*, Lessons 7, 8 and 9.

Exercises

14. The expression $\dfrac{\csc\theta}{\cot\theta\cos\theta\sec\theta}$ is equivalent to

    A. $\tan\theta$
    B. $\sec\theta$
    C. $\csc\theta$
    D. $\cos\theta$

15. The expression $\sqrt{\dfrac{\cos^2\theta+\sin^2\theta}{1+\tan^2\theta}}$, where $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$, is equivalent to

    A. $\csc\theta$
    B. $\sec\theta$
    C. $1+\tan\theta$
    D. $\cos\theta$

16. Prove that $\dfrac{\cos\theta}{1-\sin\theta}+\dfrac{\cos\theta}{1+\sin\theta}=2\sec\theta$, where $\sin\theta\neq\pm1$.

17. If θ and α are both acute angles, then find the value of $\sin(\theta - \alpha)$, where $\cos\theta = \frac{5}{13}$ and $\sin\alpha = \frac{4}{5}$.

18. If $f(\theta) = \sin(\theta + 60^\circ)$ and $t(\theta) = \cos(\theta - 30^\circ)$, prove $f(\theta) = t(\theta)$.

19. If $\cos\theta \neq 0$, then express $\dfrac{\sin^2\theta - \sin^4\theta}{\cos^4\theta}$ in terms of $\tan\theta$.

20. Prove that $\dfrac{1-2\sin^2\theta}{\sin(2\theta)+1} = \dfrac{\cos(2\theta)}{(\sin\theta+\cos\theta)^2}$.

EXERCISE 11c SOLUTIONS are at the back of this workbook.

Additional Practice

Mathematics 12: p. 297 (Questions 1-10), p. 298 (Questions 7-9)
Mathematics: Principles and Process 12: p. 212 (Question 7), p. 265 (Questions 1-3), p. 266 (Question 2)
Holtmath 12: p. 347 (Questions 1, 2, 4a, 4b, 5a, 5b, 6a, 6b), p. 348 (Questions 1a, 1b, 1e, 2, 3a, 3d, 3f, 4, 5a, 5e, 6a, 6b, 6c, 6d, 7a)

Solving Trigonometric Equations Algebraically and Graphically

MATH FACTOR

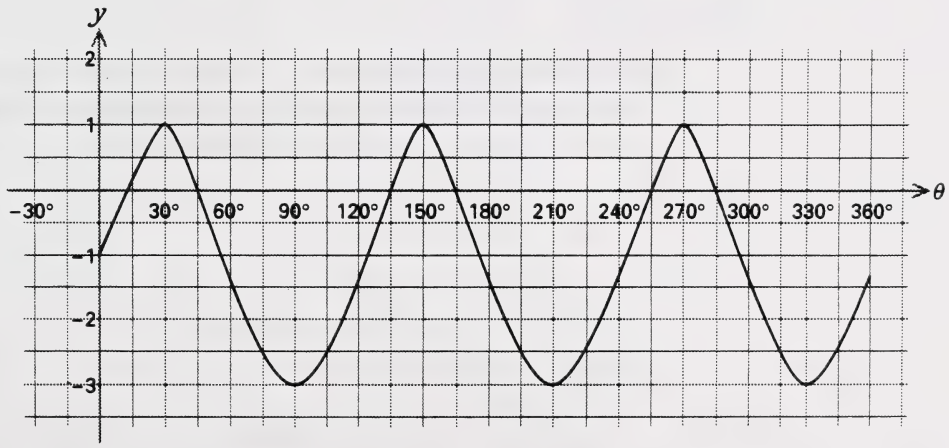
Watch Video Segment 11d.

The material in this video segment is a review of *Trigonometric and Circular Functions*, Lesson 10.

Exercises

21. Solve the equation $\sin(2\theta) + \cos\theta = 0$, where $0 \le \theta < 2\pi$.

22. The following graph represents $y = 2\sin(3\theta) - 1$, where $0° \le \theta \le 360°$.



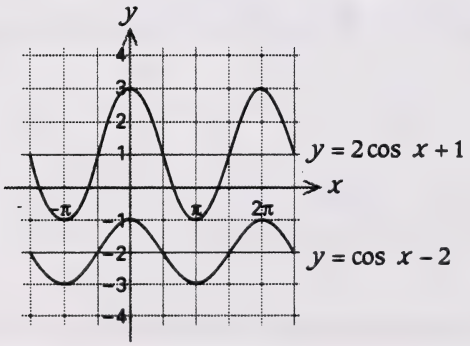
a. How many solutions are there to the equation $2\sin(3\theta) - 1 = 0$ for the given domain?

b. Find these solutions algebraically and use the graph to verify your solutions.

23. Approximate the solutions to $6\cos\theta\sin\theta - 3\cos\theta = 0$, where $0 \le \theta \le \pi$, by graphing; then algebraically verify your solutions.

24. a. Show an algebraic solution to the equation $2\cos^2 x - 3\cos x = 2$, where $0 \le x \le 2\pi$.

b. The following graphs represent $y = 2\cos x + 1$ and $y = \cos x - 2$. Explain the relationship that these two graphs have to the solution of the equation $2\cos^2 x - 3\cos x - 2 = 0$, for the domain $0 \le x \le 2\pi$.



EXERCISE 11d SOLUTIONS are at the back of this workbook.

Additional Practice

Mathematics 12: p. 297 (Question 11), p. 298 (Question 10)
Mathematics: Principles and Process 12: p. 212 (Question 8), p. 213 (Question 6), p. 265 (Questions 4, 5), p. 266 (Questions 3-5)
Holtmath 12: p. 311 (Question 9), p. 312 (Question 13), p. 314 (Question 26), p. 347 (Questions 7-9), p. 348 (Questions 8-10)

WHAT HAVE YOU LEARNED?

Return to the beginning of this lesson and check off the Learner Expectations that you have mastered. Continual review and practice will increase your understanding of basic trigonometric concepts and will enable you to apply this understanding to application and problem-solving questions. An excellent way to prepare for the final exam is to do questions from previous Alberta Education Mathematics 30 exams.

Solutions

EXERCISE 11a

1. If $\tan\theta > 0$ (positive) in quadrants 1 and 3 and $\sin\theta < 0$ (negative) in quadrants 3 and 4, then the terminal arm lies in quadrant 3.

2. $\cos\theta = \dfrac{-1}{2}$ θ is in quadrant 3.

   θ ref $= 60^\circ$

   $\theta = 240^\circ$

   $\therefore \sin\theta\cos\theta + \tan^2\theta$

   $= \sin 240^\circ \cos 240^\circ + \left(\tan 240^\circ\right)^2$

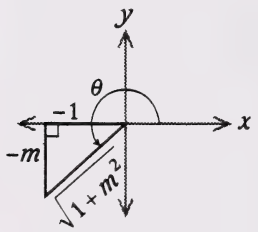
   $= \left(\dfrac{-\sqrt{3}}{2} \times \dfrac{-1}{2}\right) + \left(\sqrt{3}\right)^2$

   $= \dfrac{\sqrt{3}}{4} + 3$

   $= \dfrac{\sqrt{3}+12}{4}$

3. $\tan\theta = m$

   $\sin\theta$ is negative in quadrant 3 and quadrant 4.
   $\tan\theta$ is positive in quadrant 3 and quadrant 1.
   Therefore, m is in quadrant 3.



$r = \sqrt{1+m^2}$

$\csc\theta = \dfrac{r}{y}$

$= -\dfrac{\sqrt{1+m^2}}{m}$

4. One radian is a central angle of a circle subtended by an arc whose length is equal to the radius of the circle.

5. a. $\dfrac{3\pi}{4} \times \dfrac{180^\circ}{\pi} = 135^\circ$

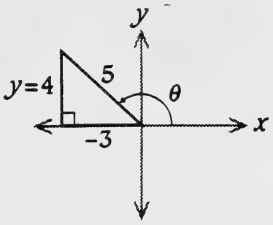
   b. $2.3 \times \dfrac{180^\circ}{\pi} \doteq 131.8^\circ$

6. a. $270^\circ \times \dfrac{\pi}{180^\circ} = \dfrac{3\pi}{2}$ rad

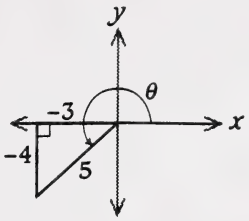
   b. $320^\circ \times \dfrac{\pi}{180^\circ} \doteq 5.6$ rad

7. $\sec\theta = \frac{-5}{3}$, $\cos\theta = \frac{-3}{5}$

Quadrant 2



Quadrant 3



a. $\sin\theta = \frac{4}{5}$ or $\sin\theta = \frac{-4}{5}$

b. θ ref : 53.1°, 0.9 rad
$\theta = 180^\circ - 53.1^\circ \doteq 126.9^\circ$
$\theta = \pi - 0.9 \doteq 2.2$ rad

or

θ ref : 53.1°, 0.9 rad
$\theta = 180^\circ + 53.1^\circ = 233.1^\circ$
$\theta = \pi + 0.9 \doteq 4.1$ rad

8. a. $\frac{-13\pi}{6}$ has a terminal arm in quadrant 4 and a reference angle of $\frac{\pi}{6}$.

$\therefore \cos\left(\frac{-13\pi}{6}\right) = \frac{\sqrt{3}}{2}$

b. $\frac{25\pi}{3}$ has a terminal arm in quadrant 1 and a reference angle of $\frac{\pi}{3}$.

$\cot\left(\frac{25\pi}{3}\right) = \cot\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$

9. $\sin^2\left(\frac{5\pi}{3}\right) \times \cos^2\left(\frac{-\pi}{4}\right) - \cot\left(\frac{7\pi}{6}\right) \times \left(-\sec\frac{\pi}{6}\right)$

$= \left(\frac{-\sqrt{3}}{2}\right)^2 \times \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\sqrt{3}\right) \times \left(\frac{-2}{\sqrt{3}}\right)$

$= \left(\frac{3}{4} \times \frac{1}{2}\right) - \left(\frac{\sqrt{3}}{1} \times \frac{-2}{\sqrt{3}}\right)$

$= \frac{3}{8} - (-2)$

$= 2\frac{3}{8}$ or $\frac{19}{8}$

EXERCISE 11b

10. a. The range of $f(x) = -7\sin(3x) + 6$ is $-1 \le f(x) \le 13$.

    b. The range of $f(x) = -k\cos(bx) + d$ is $-|k| + d \le f(x) \le |k| + d$.

11. $y = \frac{1}{3}\cos kx$

$$\frac{2\pi}{k} = \frac{2\pi}{3}$$
$$k = 3$$

12. $y = -3\cos(3\theta)$, where $0 \le \theta < 2\pi$

$-3\cos(3\theta) = 0$ $\qquad 0 \le 3\theta < 6\pi$

$\cos(3\theta) = 0$ $\qquad 0 \le \alpha < 6\pi$

Let $\alpha = 3\theta$.

$\alpha: \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}\right\}$

$\theta: \left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\}$

13. a. 16 s

    b. 15.5 m

    c. 1.5 m

    d. 7 m

    e. 13.5 m after 10 s
    8.5 m after 12 s

    f. $\frac{2\pi \text{ rad}}{16 \text{ s}} = \frac{\pi}{8}$ rad/s

    g. $h(t) = a\sin b(t + c) + d$

$$a = \frac{15.5 - 1.5}{2} = 7$$
$$16 = \frac{2\pi}{b}$$
$$b = \frac{2\pi}{16} = \frac{\pi}{8}$$
$$c = -4$$
$$d = 8.5$$
$$\therefore h(t) = 7\sin\frac{\pi}{8}(t - 4) + 8.5$$

h. $h(40) = 7\sin\frac{\pi}{8}(40-4)+8.5$

$= 7\sin\left(\frac{\pi}{8}\times 36\right)+8.5$

$= 7\left(\sin\frac{9\pi}{2}\right)+8.5$

$= 15.5 \text{ m}$

$h(64) = 7\sin\frac{\pi}{8}(64-4)+8.5$

$= 7\sin\left(\frac{\pi}{8}\times 60\right)+8.5$

$= 7\left(\sin\frac{15\pi}{2}\right)+8.5$

$= 1.5 \text{ m}$

EXERCISE 11c

14. B

$$\frac{\csc\theta}{\cot\theta\cos\theta\sec\theta} = \frac{\frac{1}{\sin\theta}}{\frac{\cos\theta}{\sin\theta}}$$

$$= \frac{1}{\cos\theta}$$

$$= \sec\theta$$

15. D

$$\sqrt{\frac{\cos^2\theta+\sin^2\theta}{1+\tan^2\theta}} = \sqrt{\frac{1}{\sec^2\theta}}$$

$$= \frac{1}{\sec\theta}$$

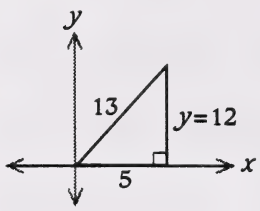
$$= \cos\theta$$

16.

L.S.	R.S.
$\frac{\cos\theta}{1-\sin\theta}+\frac{\cos\theta}{1+\sin\theta}$	$2\sec\theta$
$=\frac{\cos\theta(1+\sin\theta)+\cos\theta(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$	
$=\frac{\cos\theta+\cos\theta\sin\theta+\cos\theta-\cos\theta\sin\theta}{1-\sin^2\theta}$	
$=\frac{2\cos\theta}{\cos^2\theta}$	
$=\frac{2}{\cos\theta}$	
$=2\sec\theta$	

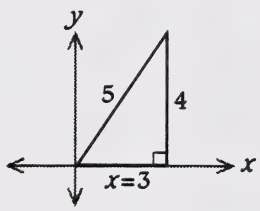
L.S. = R.S.

17. $\cos\theta=\frac{5}{13}$ quadrant 1



$\sin\theta=\frac{12}{13}$

$\sin\alpha=\frac{4}{5}$ quadrant 1



$\cos\alpha=\frac{3}{5}$

$\sin(\theta-\alpha)=\sin\theta\cos\alpha-\cos\theta\sin\alpha$

$=\frac{12}{13}\times\frac{3}{5}-\frac{5}{13}\times\frac{4}{5}$

$=\frac{36}{65}-\frac{20}{65}$

$=\frac{16}{65}$

18.

L.S.	R.S.
$\sin(\sin\theta+60^\circ)$	$\cos(\theta-30^\circ)$
$=\sin\theta\cos 60^\circ+\cos\theta\sin 60^\circ$	$=\cos\theta\cos 30^\circ+\sin\theta\sin 30^\circ$
$=\sin\theta\times\frac{1}{2}+\cos\theta\times\frac{\sqrt{3}}{2}$	$=\cos\theta\times\frac{\sqrt{3}}{2}+\sin\theta\times\frac{1}{2}$
$=\frac{1}{2}\sin\theta+\frac{\sqrt{3}}{2}\cos\theta$	$=\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta$

L.S. = R.S.

19. $\dfrac{\sin^2\theta - \sin^4\theta}{\cos^4\theta} = \dfrac{\sin^2\theta\left(1-\sin^2\theta\right)}{\cos^4\theta}$

$= \dfrac{\sin^2\theta\cos^2\theta}{\cos^4\theta}$

$= \dfrac{\sin^2\theta}{\cos^2\theta}$

$= \tan^2\theta$

20.

L.S.	R.S.
$\dfrac{1-2\sin^2\theta}{\sin(2\theta)+1}$	$\dfrac{\cos(2\theta)}{(\sin\theta+\cos\theta)^2}$
	$= \dfrac{\cos^2\theta-\sin^2\theta}{\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta}$
	$= \dfrac{\left(1-\sin^2\theta\right)-\sin^2\theta}{1+2\sin\theta\cos\theta}$
	$= \dfrac{1-2\sin^2\theta}{1+\sin(2\theta)}$

L.S. = R.S.

EXERCISE 11d

21. $\sin(2\theta)+\cos\theta = 0$, where $0 \le \theta < 2\pi$

$2\sin\theta\cos\theta+\cos\theta = 0$

$\cos\theta(2\sin\theta+1) = 0$

$\cos\theta = 0$ or $2\sin\theta+1 = 0$

$\sin\theta = \dfrac{-1}{2}$

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ $\theta = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

$\theta: \left\{\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}\right\}$

22. a. $2\sin(3\theta)-1 = 0$

There are six θ-intercepts; therefore, there are six solutions to the equation.

b. $y = 2\sin(3\theta) - 1 \quad 0° \le \theta \le 360°$

$2\sin(3\theta) - 1 = 0 \quad 0° \le 3\theta \le 1080°$ (adjust domain)

$\sin(3\theta) = \frac{1}{2}$

Let $\alpha = 3\theta$

$\sin\alpha = \frac{1}{2}$, where $0° \le \alpha \le 1080°$

α ref $= 30°$

$\alpha = \{30°, 150°, 390°, 510°, 750°, 870°\}$

$\theta = \{10°, 50°, 130°, 170°, 250°, 290°\}$

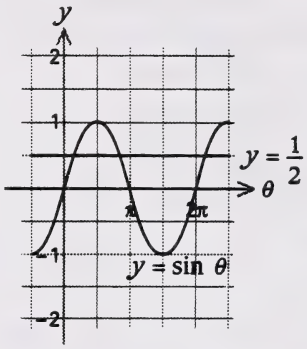
These solutions are the same as the θ-intercepts of the graph.

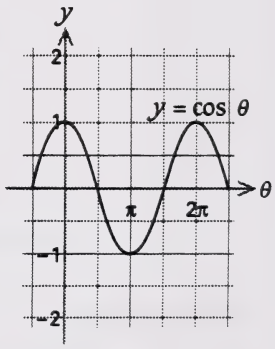
23. $6\cos\theta\sin\theta - 3\cos\theta = 0; \quad 0 \le \theta \le \pi$

$3\cos\theta(2\sin\theta - 1) = 0$

$3\cos\theta = 0$ or $2\sin\theta - 1 = 0$

$\cos\theta = 0$ $\sin\theta = \frac{1}{2}$





The solutions are the θ-intercepts of $y = \cos\theta$, and the points of intersection of $y = \sin\theta$ and $y = \frac{1}{2}$ for the stated domain.

$\therefore \theta = \left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}\right\}$

Algebraic solution:

$\cos\theta = 0$ or $\sin\theta = \frac{1}{2}$ (from previous)

$\theta = \frac{\pi}{2}$ or $\theta = \frac{\pi}{6}$ (quadrant 1) and $\frac{5\pi}{6}$ (quadrant 2)

$\therefore \theta = \left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}\right\}$

24. a. $2\cos^2 x - 3\cos x = 2$, where $0 \le x \le 2\pi$

$$2\cos^2 x - 3\cos x - 2 = 0$$
$$(2\cos x + 1)(\cos x - 2) = 0$$
$$2\cos x + 1 = 0 \quad \text{or} \quad \cos x - 2 = 0$$
$$\cos x = \frac{-1}{2} \qquad \cos x = 2$$
$$x \text{ ref} = \frac{\pi}{3} \qquad \text{no solution } (-1 \le \cos x \le 1)$$
$$x = \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$$

b. Since the corresponding functions of the factored equation were graphed, the x-intercepts of the graph $y = 2\cos x + 1$ are the solutions to the equation:

$$x = \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$$

Since the graph of $y = \cos x - 2$ does not intercept the x- axis, there are no solutions for the equation $\cos x - 2 = 0$.